For Reference

NOT TO BE TAKEN FROM THIS ROOM

For Reference

NOT TO BE TAKEN FROM THIS ROOM

Ex libris
universitatis
albertaensis

QUAECUMQUE VERA

The University of Alberta
Printing Department
Edmonton, Alberta

ON PROXIMITY SPACES

by

©

BRIAN D. WARRACK

A THESIS
SUBMITTED TO THE FACULTY OF GRADUATE STUDIES
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE
OF MASTER OF SCIENCE

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF ALBERTA
EDMONTON, ALBERTA

OCTOBER, 1967

UNIVERSITY OF ALBERTA

FACULTY OF GRADUATE STUDIES

The undersigned certify that they have read and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled "ON PROXIMITY SPACES", submitted by BRIAN D. WARRACK in partial fulfillment of the requirements for the degree of Master of Science.

ABSTRACT

This thesis represents an attempt on the part of the author to present a needed systematic exposition of topics occurring in the mainstream of proximity spaces. Shortly after Efremovič introduced proximity spaces in 1951, Smirnov compiled a survey of the subject in Russian. Since then, numerous papers have appeared introducing such new ideas as clusters into proximity theory. Clusters play a fundamental role in the development of proximity spaces, and the relationship found by Warrack and Naimpally between clusters and ultrafilters simplifies the theory considerably. Moreover, advantage of hindsight has been taken to simplify several of the proofs in the first two chapters.

In Chapter III, the concept of a Cauchy cluster is introduced and is used to present a new approach to the completion of a uniform space.

In order to minimize the number of discontinuities, no references appear in the main body of the thesis. Instead, these references are collected in the notes at the end of each chapter, as well as in the historical background. A reasonably complete bibliography of the literature on proximity spaces is also included.

ACKNOWLEDGEMENTS

I would like to express my sincere gratitude to my supervisor, Dr. S. A. Naimpally, for his guidance, encouragement and contagious enthusiasm throughout the preparation of this thesis. I am also deeply indebted to the University of Alberta and the National Research Council for providing the necessary financial assistance.

HISTORICAL BACKGROUND

The germ of the theory of proximity spaces showed itself as early as 1908 at the mathematical congress in Bologna, when Riesz [85] discussed various ideas in his "theory of enchainment" which have today become the basic concepts of the theory. The subject was essentially rediscovered in the early 1950's by Efremovič [17, 18] when he axiomatically characterized the proximity relation "A is near B" for subsets A,B of any set X . The set X together with this relation was called an infinitesimal (proximity) space, and is a natural generalization of a metric space and of a topological group. A decade earlier a study was made by each of Krishna Murti [47], Wallace [103, 104] and Szymański [101] concerning the use of "separation of sets" as the primitive concept. In each case similar, but weaker, axioms than those of Efremovič were used. Efremovič later used proximity neighbourhoods to obtain an equivalent set of axioms for a proximity space and thereby an alternative approach to the theory.

Defining the closure of a subset A of X to be the collection of all points of X "near" A , Efremovič [18] showed that a topology can be introduced on a proximity space and that we obtain in fact a completely regular (and hence uniformizable) space. He showed further that every completely regular space X can be turned into a proximity space with the help of Urysohn's function: namely, $A \not\delta B$ iff there exists an continuous function f mapping X onto [0,1] such that $f(A) = 0$, $f(B) = 1$. Smirnov [88] subsequently proved that every completely regular space has a maximal associated proximity space, and that

it has a minimal associated proximity space if and only if it is locally compact.

More recently, Goetz [26] has introduced a notion similar to that of proximity in a Frechet L-space to obtain what he terms a $\underset{\sim}{U}\,\underset{\sim}{L}$ - space. This is accomplished by defining a nearness relation on the set of all sequences from a Fréchet L-space. However, as opposed to the complete regularity of a proximity space, a $\underset{\sim}{U}\,\underset{\sim}{L}$ - space need not even be a topological space. Poljakov [83] and Goetz [27] have since carried out further investigations in this area, and discuss the connection between this notion and the proximity of Efremovič. Švarc [19] had earlier introduced a nearness relation on the set of all nets from a given space.

In order to study mappings of one proximity space to another, it was natural for Efremovič to introduce the concept of a proximity mapping. Defined to be a mapping which preserves the proximity of sets, this is a natural generalization of a continuous mapping in a topological space since the latter preserves the proximity of points. It is readily verified that a proximity mapping between two proximity spaces is continuous with respect to the induced topologies. Pervin [78] later revealed that the converse holds if the domain proximity space is equinormal.

In 1952, Smirnov [88] pursued extensions of proximity spaces and in particular answered Alexandroff's query: "Which topological spaces admit a proximity relation compatible with the given topology?". He discovered the connection between the Hausdorff compatification of

a Tychonoff space and the compatible proximity relation, showing that "a topological space admits a compatible proximity relation if and only if it is a subspace of a compact Hausdorff space". Using the "ends" of Alexandroff [A], Smirnov [88] obtained the compactification of a proximity space X by identifying each point $x \in X$ with the end consisting of all proximity neighbourhoods of x and showing the compactification of X to be the set of all ends in X .

The concept of a cluster, the analogue in a proximity space of an ultrafilter, was introduced by Leader [48] and provides an alternative approach to many proximity problems. In particular, Leader [48] obtained the compactification of a proximity space X to be the family of all clusters from X . Since the Smirnov compactification is unique, it is evident that a one-to-one correspondence exists between clusters and ends. In fact, Leader [50] proves (without resorting to compactification theory) that clusters and ends are dual classes.

Császár and Mrówka [13] proved a stronger result regarding the compactification of a proximity space, namely: Compactification can be effected preserving the proximity weight. This gives rise to the following interesting metrization theorem: A proximity space of proximity weight $\aleph_0$ is metrizable. Additional solutions to the metrization problem were offered earlier by Švarc [19], who used the "sequence-uniformity" method, and by Ramm and Švarc [84] using uniform proximity covers. More recently, Smirnov [95] has used both the pseudo-metric and proximity neighbourhood approaches in deriving necessary and sufficient conditions for metrizability.

Using the alternate set of axioms for uniform spaces involving uniform coverings, Smirnov [88] showed that every proximity (p-) equivalence class of uniform structures contains a coarsest or minimal member, which is also the unique totally bounded structure of the class. Moreover, this is the only structure on which the proximity space is continuable to the compact proximity extension. Finally, he showed there is an isomorphism between the partially ordered set of proximities on a given completely regular space and the partially ordered set of all its compactifications, which reduces the theory of proximity spaces to that of compactifications.

In 1959, Gál [25] continued the pursuit of the relationships between uniform structures and proximities, and proved there is a natural order-preserving one-to-one correspondence between totally bounded (precompact) structures and proximity relations. Gál also showed that there is a one-to-one correspondence between the compactification of a uniformizable space and the totally bounded uniform structures which are compatible with its topology. Again, this yields a one-to-one map between the Hausdorff compactifications and the separated structures. He also presents the following basic theorem: "Two subsets of X are far with respect to the proximity relation corresponding to the totally bounded structure $\underset{\sim}{U}$ if and only if they have disjoint closures in the compactification of X relative to $\underset{\sim}{U}$".

In the same year, Alfsen and Fenstad [4] paralleled the work done by Gál and treated many of the problems of Efremovič and Smirnov in the framework of Weil's uniform structure. Using maximal regular

filters Alfsen and Fenstad perform completion and compactification, showing that there exists a maximal completion amongst all the completions determined by the structures of a given proximity equivalence class. This naturally raises the question as to whether there always exists a minimal completion, which is equivalent to asking if there exists a finest (or maximal) uniform structure in a given proximity equivalence class. This question is answered affirmatively by Smirnov [88] for the case in which the proximity space is metrizable. That the answer is in general negative was first shown in 1961 by Dowker [16]. He unveiled an example of a proximity space (a product of two infinite spaces with the product proximity structure) which has no finest uniform structure inducing its proximity.

The key concept in any completion theory for proximity spaces is that of "small" sets, which can be introduced by means of pseudometrics, uniform structures or uniform coverings. Leader [49] uses the first device to define a local cluster, while Warrack and Naimpally [106] use the second to define a Cauchy cluster. They obtain the completion to be the family of all local (resp. Cauchy) clusters in X, a subspace of the family of all clusters which Leader showed to be the Smirnov compactification. In [89,90], Smirnov uses uniform δ-coverings to introduce the concept of a complete uniform space and proves that every proximity space admits a minimal completion of this kind. Thus the general existence of a minimal completion is established, but with the sacrifice of the one-to-one correspondence between uniform structures and completions.

This is regained in the work of Alfsen and Njåstad [6], who in 1963 gave yet another example of a p-equivalence class lacking a finest uniform structure. (Further examples are to be found in Leader [51], Fenstad [21] and Isbell [39].) This then led to the notion of generalized uniform structures by replacing the "intersection" axiom by a less restrictive one. They prove that for generalized uniform structures the answer to Smirnov's problem is affirmative, and give an explicit characterization of those generalized uniform structures which occur as the finest member of their respective p-equivalence classes. These are called the total structures, and it is shown that the class of total generalized uniform structures comprises all ordinary metrizable (or pseudo-metrizable) uniform structures. They obtain a general characterization of proximity continuity by means of uniform continuity, and as particular consequences show Efremovič's result that metric uniform continuity is equivalent to metric proximity continuity, and Alfsen and Fenstad's result that totally bounded uniform continuity is equivalent to proximity continuity. They prove that every generalized uniform space can be completed and establish a one-to-one correspondence between generalized uniform structures and proximity space completions. In particular, the minimal completion of a proximity space is obtained by completion of the finest generalized structure compatible with the proximity structure.

In the same year, Njåstad [72] further developed the concept of a generalized uniform structure. He first notes that the collection of all generalized uniform structures on a set and the collection of all proximity structures on a set form complete lattices when ordered by the

relations "finer - coarser", and that lattice sums and products are compatible unlike the case in which the usual uniform structures are considered. Moreover, he establishes the existence and compatibility of initial (final) generalized uniform structures and initial (final) proximity structures. He proves that if the uniformity is replaced by the associated proximity, then uniform convergence implies convergence in proximity, a notion introduced by Leader [49]. Convergence in proximity implies uniform convergence if the associated total uniformity is used in place of the proximity. Moreover, for a generalized sequence (or net) of functions with a linearly ordered index set, the two forms of convergence are equivalent. In 1965, Hursch [33] formulated a new concept, height, to help clarify the order structure of p-equivalence classes of uniformities.

As previously mentioned, some authors have worked with weaker axioms than those of Efremovič, enabling them to introduce an arbitrary topology on the underlying set. With such generalized proximities as quasi-proximity, paraproximity, pseudo-proximity and local proximity already existing in the literature, one almost wonders if a generalized proximity relation may be defined for each prefix which can possibly be attached to the word "proximity". In 1963, both Pervin [76] and Leader [52] independently studied generalizations of Efremovič's original set of axioms. Pervin neglected the symmetry condition, obtaining what he called a quasi-proximity space. As well as omitting the symmetry condition, Leader also used a weakened form of the "Strong Axiom" to arrive at a topological d-space. It was shown that every topological space can be derived from a generalized proximity space (X,δ) of

either form by defining the binary relation δ as follows: $A \delta B$ iff $A \cap \overline{B} \neq \emptyset$. Conversely, every quasi-proximity or topological d-space becomes a topological space if the closure operator is defined by $c(A) = \{x : \{x\} \delta A\}$. Lodato [56] later added symmetry to Leader's set of axioms to obtain a symmetric binary relation which we shall refer to as a Lodato proximity. He found that every set with a Lodato proximity defined on it satisfies the R_o axiom (i.e. every open set contains the closure of each of its points), and that given any R_o space we obtain a Lodato proximity compatible with the given topology if we define $A \delta B$ iff $\overline{A} \cap \overline{B} \neq \emptyset$.

Around the same time, Hayashi [30] introduced the notion of "paraproximity" by replacing the word "finite" by "arbitrary", and thereby strengthening Efremovič's "union" axiom to read: For an arbitrary index set Λ , $(\bigcup_{\lambda \in \Lambda} A_\lambda) \delta B$ iff $A_\mu \delta B$ for some $\mu \in \Lambda$. He showed that a paraproximity space is a completely normal space X if we define G to be an open set if and only if $G \not\delta (X - G)$. A completely normal space becomes a paraproximity space if we define $A \delta B$ iff $A \cap \overline{B} \neq \emptyset$. Hayashi [31] also discussed a generalized proximity space, which he called a pseudo-proximity space, with even weaker axioms than those considered by Pervin or Leader. He proved that every such space can be topologized and that every topological space induces a pseudo-proximity.

Recently, Leader [54] has defined a local proximity space, in which both "proximity" and "boundedness" are taken as primitive

terms. The proximity spaces of Efremovič are the special cases in which all subsets are bounded. Just as every proximity space can be embedded as a dense subset of a compact Hausdorff space, it is shown that every local proximity space can be embedded as a dense subset of a locally compact Hausdorff space. Leader shows that every proximity space (X,δ) with its proximity relation localized with respect to any free, regular filter from (X,δ) gives rise to a local proximity space. Conversely, every local proximity space arises from the localization of some proximity relation.

TABLE OF CONTENTS

Page

ABSTRACT . . . . (i)
ACKNOWLEDGEMENTS . . . . (ii)
HISTORICAL BACKGROUND . . . . (iii)

CHAPTER I: BASIC PROPERTIES

§1. Introduction 1
§2. Topology induced by a proximity 6
§3. Alternate description of proximity . . . 14
§4. Subspaces. Product and quotient proximity spaces 22
Notes 32

CHAPTER II: COMPACTIFICATIONS OF PROXIMITY SPACES

§5. Clusters and ultrafilters 33
§6. Duality in proximity spaces 45
§7. Smirnov compactification 52
§8. Proximity weight and compactification. . 64
§9. Local proximity spaces 73
Notes 85

CHAPTER III: PROXIMITY AND UNIFORMITY

§10. Proximity induced by a uniformity . . . 87
§11. Completion of a uniform space by Cauchy clusters 93
Notes 98

GENERAL REFERENCES 99
BIBLIOGRAPHY FOR PROXIMITY SPACES 100

CHAPTER I

BASIC PROPERTIES

§1. Introduction

In a topological space X, the topology is determined by the closure axioms concerning the relation "x is a closure point of $A \subset X$", which were given by Kuratowski. When x is a closure point of A, we may say that "x is near A". In terms of this nearness relation, a continuous function $f:X \rightarrow Y$ may be described as one exhibiting the property: if x is near A then $f(x)$ is near $f(A)$. This suggests axiomatizing the relation "A is near B" for subsets A and B of X. For the case in which X is a pseudo-metric space with pseudo-metric d, this nearness relation can be defined in a natural way. Let $D(A,B) = \inf \{d(a,b) : a \in A, b \in B\}$. We may then define: A is near B if and only if $D(A,B) = 0$. In terms of D, the closure of a set A is $\overline{A} = \{x : D(A,x) = 0\}$. But the nearness relation so defined goes a little further. Let (Y,e) be another pseudo-metric space, E be defined in a similar manner to D, and f be a function from X to Y. Then f is uniformly continuous if and only if $D(A,B) = 0$ implies $E(f(A),f(B)) = 0$. Thus the nearness relation between the subsets is somehow connected with uniformity.

The above nearness relation (in a pseudo-metric space) satisfies the following properties, where we denote "A is near B" by $A \,\delta\, B$:

(1.1) $A \,\delta\, B$ implies $B \,\delta\, A$.

(1.2) $(A \cup B)\ \delta\ C$ iff $A\ \delta\ C$ or $B\ \delta\ C$.

(1.3) $A\ \delta\ B$ implies $A \neq \emptyset$, $B \neq \emptyset$.

(1.4) $A\ \not\delta\ B$ implies there exists a subset E such that $A\ \not\delta\ E$ and $(X - E)\ \not\delta\ B$.

(1.5) $A \cap B \neq \emptyset$ implies $A\ \delta\ B$.

In a metric space, the nearness relation also satisfies

(1.6) $x\ \delta\ y$ implies $x = y$.

(Strictly speaking one should use the notation $\{x\}\ \delta\ \{y\}$, but we shall simply write $x\ \delta\ y$.)

All of the above properties, except perhaps (1.4), are immediate consequences of the definition of the nearness relation. To verify (1.4), we note that if $A\ \not\delta\ B$ then $D(A,B) = \epsilon > 0$. If we let $E = \{x \in X : D(x,B) \leq \epsilon/2\}$, then $D(A,E) \geq \epsilon/2$, $D(X - E, B) \geq \epsilon/2$ and the result follows.

The above discussion leads to the following definition of a proximity space:

(1.7) Definition. A binary relation δ on 2^X is called a proximity on X iff δ satisfies the axioms (1.1)-(1.5). The pair (X,δ) is called a proximity space.

Proximity relations satisfying Axiom (1.6) will be referred to as "separated" (or Hausdorff) proximity relations. If a proximity is derived from a (pseudo-) metric, then it is called a (pseudo-) metric proximity.

(1.8) Remarks. The above axioms are different from, although equivalent to, the original axioms of Efremovič. The reason for writing the axioms in this way is to smoothen the transition to generalized proximity spaces. It will be shown presently that the proximity δ on X induces a topology $\tau = \tau(\delta)$ by defining the closure $\overline{A}$ of A as the set $\{x : x \,\delta\, A\}$. It will be seen that this topology is always completely regular; in fact it is always Tychonoff if Axiom (1.6), which is equivalent to the T_1-axiom, is satisfied.

Actually it would be sufficient to concern ourselves solely with separated proximity spaces, as many authors do; for if a given space fails to satisfy condition (1.6), we can instead consider the separated quotient space formed by the equivalence classes consisting of all points near to one another. However, for the sake of generality, we shall not assume a proximity space to be separated unless specified.

It is also known that if (1.2)-(1.5) and

(1.2') $$A \,\delta\, (B \cup C) \text{ iff } A \,\delta\, B \text{ or } A \,\delta\, C$$

are satisified by a relation δ', then the symmetry axiom (1.1) is equivalent to the complete regularity of the associated topology. By this we mean that $\tau(\delta')$ need not satisfy any separation axiom unless δ' is symmetric, in which case $\tau(\delta')$ is necessarily completely regular, and that every completely

regular space (X,τ) possesses a proximity δ such that $\tau = \tau(\delta)$.

Axiom (1.4) plays an important role in the theory of proximity spaces, but is omitted or replaced by a weaker condition in some generalized proximity spaces. It will, therefore, be helpful to avoid the use of this axiom as far as possible. We shall refer to this axiom as "the Strong Axiom." It should be noted that the order of the sets in the Strong Axiom is important, particularly in generalized proximity spaces in which (1.1) is not satisfied. We shall strictly observe the order, even in proximity spaces, so that the proofs can then be carried over to more general situations.

Given below is an outline of a number of examples of proximity spaces other than a (pseudo-) metric proximity space. Since some of these will be carefully taken up in later sections, the details are not verified here.

(1.9) Example. Just as discrete and indiscrete topologies can be defined on any set, we have discrete and indiscrete proximities. If we define $A \,\delta_1\, B$ iff $A \cap B \neq \phi$, then δ_1 is the discrete proximity on X . On the other hand, if $A \,\delta_2\, B$ for every pair of non-empty subsets A and B of X , then we obtain the indiscrete proximity on X .

(1.10) Example. Given a completely regular space (X,τ), we say that subsets A,B of X are functionally distinguishable iff there is a continuous function $f:X \rightarrow [0,1]$ such that $f(A) = 0$ and $f(B) = 1$. We may then define a proximity δ by

(1.11) $A \not\delta B$ iff A and B are functionally distinguishable.

For details, refer to Theorem (2.10) and Remarks (3.14).

(1.12) Example. Given a uniform space $(X, \underset{\sim}{U})$, we may define a proximity by

$A \delta B$ iff for every $U \in \underset{\sim}{U}$, one (and hence all) of the three following equivalent conditions is satisfied:

$$\text{(i)} \quad U[A] \cap B \neq \emptyset$$

$$\text{(ii)} \quad U[A] \cap U[B] \neq \emptyset$$

$$\text{(iii)} \quad (A \times B) \cap U \neq \emptyset \ .$$

There are several interesting results concerning uniformities and proximities which will be discussed in Chapter III.

(1.13) Example. If $(X, \circ, \tau)$ is a topological group and $\underset{\sim}{N}$ is the neighbourhood system of the identity, then we may define

$$A \delta_1 B \text{ iff for every } N \in \underset{\sim}{N}, \ NA \cap B \neq \emptyset \ .$$

A second proximity δ_2 may be defined by

$$A \delta_2 B \text{ iff for every } N \in \underset{\sim}{N}, \ AN \cap B \neq \emptyset \ .$$

In general the two proximities δ_1 and δ_2 differ. They coincide, however, if X is either commutative or compact.

§2. Topology Induced by a Proximity

In this section we consider the topology on X which is induced by a proximity on X, and study its elementary properties. Properties (i) and (ii) of the following lemma, which follow directly from Axioms (1.2) and (1.4) respectively, are useful in several proofs.

(2.1) **Lemma.** (i) If $A \delta B$, $A \subset C$ and $B \subset D$, then $C \delta D$. Hence X is near every non-empty subset.

(ii) If there exists an x such that $x \delta A$ and $x \delta B$, then $A \delta B$.

(2.2) **Theorem.** If a subset A of a proximity space (X,δ) is defined to be closed iff $x \delta A$ implies $x \in A$, then the collection of complements of all closed sets so defined yields a topology $\tau = \tau(\delta)$ on X.

Proof: Obviously ϕ and X are closed sets. Let $\{A_i : i \in I\}$ be an arbitrary collection of closed sets. If $x \delta \bigcap_{i \in I} A_i$ then by Lemma (2.1), $x \delta A_i$ for each $i \in I$, i.e. $x \in A_i$ for each $i \in I$ since each A_i is closed. Thus $x \in \bigcap_{i \in I} A_i$, which means $\bigcap_{i \in I} A_i$ is closed. Finally, if A_1 and A_2 are closed and $x \delta (A_1 \cup A_2)$ then by (1.2), either $x \delta A_1$ or $x \delta A_2$. But A_1 and A_2 are closed, implying that $x \in A_1$ or $x \in A_2$, i.e. $x \in (A_1 \cup A_2)$. Thus $A_1 \cup A_2$ is closed.

(2.3) **Theorem.** Let (X,δ) be a proximity space and $\tau = \tau(\delta)$. Then

the τ-closure $\overline{A}$ of a set A is given by

$$\overline{A} = \{x : x \,\delta\, A\} .$$

Proof: If $\overline{A}$ denotes the intersection of all closed sets containing A and $A^{\delta} = \{x : x \,\delta\, A\}$, then we must show that $\overline{A} = A^{\delta}$. If $x \in A^{\delta}$ then $x \,\delta\, A$. By (2.1) this implies $x \,\delta\, \overline{A}$ and, since $\overline{A}$ is closed, $x \in \overline{A}$. Thus $A^{\delta} \subset \overline{A}$. To prove the reverse inclusion it suffices to prove that A^{δ} is closed, i.e. $x \,\delta\, A^{\delta}$ implies $x \in A^{\delta}$. Assuming $x \notin A^{\delta}$, then $x \,\not\delta\, A$ so that, by the Strong Axiom, there is a set E such that $x \,\not\delta\, E$ and $(X - E) \,\not\delta\, A$. Since $A^{\delta} \subset E$ and $x \,\not\delta\, E$, it follows that $x \,\not\delta\, A^{\delta}$.

(2.4) Corollary. If G is a subset of a proximity space (X,δ), then $G \in \tau(\delta)$ iff $x \,\not\delta\, (X - G)$ for every $x \in G$.

(2.5) Corollary. If A and B are subsets of a proximity space (X,δ), then $A \,\not\delta\, B$ implies

(i) $\overline{B} \subset (X - A)$

(ii) $B \subset \text{Int}\,(X - A)$

where the closure and interior are taken with respect to $\tau(\delta)$.

Proof: (i) follows directly from (2.1). To prove (ii), we use the identity: $\text{Int}\,(X - A) = X - \overline{A}$. Then $x \notin \text{Int}\,(X - A)$ implies $x \in \overline{A}$, so that $x \,\delta\, A$ and hence $x \notin B$.

(2.6) Remarks. Theorem (2.3) is true if we omit the Symmetry Axiom (1.1) and add (1.2'). An alternative method of introducing the same topology on a proximity space (X,δ) would be to define for each subset A of X,

$$A^{\delta} = \{x : x \, \delta \, A\} \tag{2.7}$$

and show that ${}^{\delta}$ is a Kuratowski closure operator as follows:

(i) By (1.3), $x \not\delta \phi$ implies $\phi = \phi^{\delta}$.

(ii) By (1.5), $x \in A$ implies $x \, \delta \, A$, so that $A \subset A^{\delta}$.

(iii) By (1.2), $x \in (A \cup B)^{\delta}$ iff $x \, \delta \, (A \cup B)$ iff $x \, \delta \, A$ or $x \, \delta \, B$ iff $x \in A^{\delta}$ or $x \in B^{\delta}$ iff $x \in (A^{\delta} \cup B^{\delta})$. Thus $(A \cup B)^{\delta} = A^{\delta} \cup B^{\delta}$.

(iv) To prove $(A^{\delta})^{\delta} \subset A^{\delta}$, suppose $x \notin A^{\delta}$, i.e. $x \not\delta A$. Then using the Strong Axiom, there exists an E such that $x \not\delta E$ and $(X - E) \not\delta A$. Now $A^{\delta} \subset E$ and $x \not\delta E$, so that $x \not\delta A^{\delta}$ and $x \notin (A^{\delta})^{\delta}$.

(2.8) Lemma. For subsets A and B of a proximity space (X,δ),

$$A \, \delta \, B \quad \text{iff} \quad \bar{A} \, \delta \, \bar{B} ,$$

where the closure is taken with respect to $\tau(\delta)$.

Proof: Necessity is a trivial consequence of (2.1). To prove sufficiency, suppose $A \not\delta B$. Then by the Strong Axiom, there exists an E such that $A \not\delta E$ and $(X - E) \not\delta B$. By

Corollary (2.5), $\overline{B} \subset E$ and by (2.1), $A \not\delta E$ implies $A \not\delta \overline{B}$. It then follows from the Symmetry Axiom that $\overline{A} \not\delta \overline{B}$.

(2.9) Definition. If on a set X there is a topology τ and a proximity δ such that $\tau = \tau(\delta)$, then τ and δ are said to be compatible.

(2.10) Theorem. If (X,τ) is a completely regular space, then the proximity δ defined by (1.11), namely $A \not\delta B$ iff A and B are functionally distinguishable, is compatible with τ. If (X,τ) is a Tychonoff space, then δ is separated.

Proof: To show that δ is a proximity it suffices to prove that the Strong Axiom is satisfied, since the other proximity axioms are easily verified. Suppose $A \not\delta B$ and let f be a continuous function from X to $[0,1]$ such that $f(A) = 0$ and $f(B) = 1$. Let $E = \{x \in X : 1/2 \leq f(x) \leq 1\}$. Then $A \not\delta E$ and $(X - E) \not\delta B$. For instance, if g denotes the self-mapping of $[0,1]$ defined by

$$\begin{aligned} g(y) &= 2y \qquad 0 \leq y \leq 1/2 \\ &= 1 \qquad 1/2 \leq y \leq 1 \end{aligned}$$

then g (and hence gf) is continuous and $gf : X \to [0,1]$, is such that $gf(A) = 0$ and $gf(E) = 1$.

To see that δ is separated if (X,τ) is Tychonoff, we note that if $x \neq y$ then $x \notin \overline{y}$ since (X,τ) is T_1. Using Tychonoff's characterization of a completely regular space,

we are assured that x and $\bar{y}$ are functionally distinguishable, implying that $x \not\delta y$.

We now show that $\tau = \tau(\delta)$. Let $G \in \tau$ and $x \in G$. Then x is not in the closed set $X - G$, and so there exists a continuous function $f : X \to [0,1]$ such that $f(x) = 0$ and $f(X - G) = 1$, i.e. $x \not\delta (X - G)$. Then by (2.4) , $G \in \tau(\delta)$. Conversely, if $G \in \tau(\delta)$ and $x \in G$, then $x \not\delta (X - G)$. Hence there exists a continuous function $f : X \to [0,1]$ such that $f(x) = 0$ and $f(X - G) = 1$. Then $f^{-1}([0,1/2))$ is an open τ-neighbourhood of x contained in G . Therefore we have $G \in \tau$.

Thus we know that on every completely regular (Tychonoff) space, we can define a compatible (separated) proximity. The converse of Theorem (2.10) is also true; that is, the topology $\tau(\delta)$ induced by a (separated) proximity δ is always (Tychonoff) completely regular (see Theorem (3.14)).

We now show that in a T_4-(normal T_1-) space, there is another way of defining a compatible proximity.

(2.11) Theorem. In a normal T_1-space (X,τ) ,

$$A \,\delta\, B \quad \text{iff} \quad \bar{A} \cap \bar{B} \neq \emptyset \tag{2.12}$$

defines a compatible proximity.

Proof: That (2.12) defines a proximity follows from

Theorem (2.10) and the fact that, in a normal space, $\bar{A} \cap \bar{B} = \emptyset$ iff $\bar{A}$ and $\bar{B}$ are functionally distinguishable. (Note that for this part, the T_1-axiom is unnecessary.) To show that $\tau = \tau(\delta)$, we observe that a normal T_1-space is Tychonoff and use Theorem (2.10).

(2.13) Corollary. In a normal space (X,τ), the proximities defined by (1.11) and by (2.12) are equivalent.

(2.14) Corollary. If a completely regular space (X,τ) has a compatible proximity δ defined by (2.12), then X is normal.

Proof: If A and B are disjoint closed sets, then $A \not\delta B$. By the Strong Axiom, there exists an E such that $A \not\delta E$ and $(X - E) \not\delta B$. From (2.5) (ii), we then have $A \subset \text{Int}\,(X - E)$ and $B \subset \text{Int}\, E$. Since $\text{Int}\, E \cap \text{Int}\,(X - E) = \emptyset$, X is normal.

In the same way as the class of topologies on a given set can be partially ordered by inclusion, i.e. $\tau_1 \leq \tau_2$ iff $\tau_1 \subset \tau_2$, one can define a partial order on a set X in the following manner:

(2.15) Definition. If δ_1 and δ_2 are two proximities on a set X, we define

$$\delta_1 \geq \delta_2 \quad \text{iff} \quad A\,\delta_1\,B \text{ implies } A\,\delta_2\,B. \tag{2.16}$$

The following theorem shows that a finer proximity structure induces a finer topology.

(2.17) Theorem. (a) Let δ_1 and δ_2 be two proximities defined on a set X. Then $\delta_1 \leq \delta_2$ implies $\tau(\delta_1) \leq \tau(\delta_2)$.

(b) Let τ_1 and τ_2 be two completely regular topologies on X, and let δ_1 and δ_2 be the proximities on X defined by (1.11) with respect to τ_1 and τ_2 respectively. Then $\tau_1 \leq \tau_2$ implies $\delta_1 \leq \delta_2$.

Proof: (a) Suppose $A \in \tau(\delta_1)$. Then by (2.4), $x \not\delta_1 (X - A)$ for each $x \in A$. Since $\delta_1 \leq \delta_2$, $x \not\delta_2 (X - A)$ for each $x \in A$. Thus $A \in \tau(\delta_2)$, from which we conclude that $\tau(\delta_1) \subset \tau(\delta_2)$, i.e. $\tau(\delta_1) \leq \tau(\delta_2)$.

(b) If $A \not\delta_1 B$ then there exists a τ_1-continuous function $f : (X,\tau_1) \to [0,1]$ such that $f(A) = 0$ and $f(B) = 1$. Since $\tau_1 \leq \tau_2$, f is also a τ_2-continuous function, showing that $A \not\delta_2 B$. Hence $\delta_1 \leq \delta_2$.

(2.18) Remarks. For a stronger form of Theorem 2.17 (b), the reader is referred to Remarks (3.15).

It is sometimes said that a proximity is a finer structure on a set than a topology, since a topological space may have different compatible proximities. For example, let X be the real line with the usual topology and let δ_1 be defined by $A \,\delta_1\, B$ iff $D(A,B) = 0$, where $D(A,B) = \inf \{|a-b| : a \in A, b \in B\}$. Let δ_2 be defined by

(2.12). Then both δ_1 and δ_2 are compatible with the topology of X. However, $A = \{n : n \in N\}$ and $B = \{n - \frac{1}{n} : n \in N\}$ are such that $A \,\delta_1\, B$ but $A \,\not\!\delta_2\, B$.

§3. Alternate Description of Proximity

In a uniform space $(X,\underset{\sim}{U})$, a subset B may be said to be a uniform neighbourhood of A iff there is an entourage $U \in \underset{\sim}{U}$ such that $U[A] \subset B$. An analogous concept, that of a δ-neighbourhood, can be introduced in a proximity space and furnishes an alternative approach to the study of proximity spaces. This concept of a δ-neighbourhood is not only useful in the theory of proximity spaces, but also is, in a sense, dual to the concept of proximity. This duality will be studied in detail in Section 6.

(3.1) Definition. A subset B of a proximity space (X,δ) is a δ-neighbourhood of A (in symbols $A \ll B$) iff $A \not\delta (X - B)$.

The second part of the following lemma (which is a strengthened form of (2.5)) justifies the term "δ-neighbourhood".

(3.2) Lemma. Let (X,δ) be a proximity space and let $\overline{A}$ and $\operatorname{Int} A$ denote, respectively, the closure and interior of A in $\tau(\delta)$. Then

(i) $A \ll B$ implies $\overline{A} \ll B$

(ii) $A \ll B$ implies $A \ll \operatorname{Int} B$.

Therefore $A \subset \operatorname{Int} B$, showing that a δ-neighbourhood is a topological neighbourhood.

Proof: (i) Using (2.8), $A \not\delta (X - B)$ implies $\overline{A} \not\delta (X - B)$, i.e. $\overline{A} \ll B$.

(ii) $A \not\delta (X - B)$ implies $A \not\delta \overline{(X - B)}$. Equivalently,

$A \not\delta (X - \text{Int } B)$, i.e. $A \ll \text{Int } B$.

The following is an alternate way of stating the Strong Axiom and is employed by many authors.

(3.3) Lemma. The Strong Axiom (1.4) is equivalent to

(3.4) $A \not\delta B$ implies there exist subsets C and D such that $A \not\delta (X - C)$, $B \not\delta (X - D)$ and $C \not\delta D$.

Proof: To prove (3.4) implies (1.4), we note that if $C \not\delta D$, then $C \subset (X - D)$. Setting $E = X - C$, we have $A \not\delta E$ and $(X - E) \not\delta B$. On the other hand, suppose (1.4) holds. Then $A \not\delta B$ implies there is an E_1 such that $A \not\delta E_1$ and $(X - E_1) \not\delta B$. Moreover, there exists an E_2 such that $A \not\delta E_2$ and $(X - E_2) \not\delta E_1$. Now let $C = X - E_2$ and $D = E_2$. Then $C \subset X - E_1$ and it follows that $C \not\delta D$.

(3.5) Corollary. $A \not\delta B$ implies there exist subsets C and D such that $A \ll C$, $B \ll D$ and $C \not\delta D$. Consequently the topology $\tau(\delta)$ of a separated proximity space (X,δ) is Hausdorff.

(3.6) Lemma. Let δ be a compatible proximity on a completely regular space (X,τ) . If A is compact, B is closed and $A \cap B = \phi$, then $A \not\delta B$.

Proof. For each $a \in A$, $a \not\delta B$. From (3.2) and (3.5), we know there exists an open neighbourhood N_a of a such that $N_a \not\delta B$. Now $\{N_a : a \in A\}$ is an open cover of the compact set A, and so there is a finite subcover $\{N_{a_i} : i = 1,\ldots,n\}$. By (1.2), $N \not\delta B$ where $N = \bigcup_{i=1}^{n} N_{a_i}$. But $A \subset N$, implying that $A \not\delta B$.

It is well-known that a compact Hausdorff space has a unique compatible uniformity. A similar theorem is true in proximity spaces.

(3.7) Theorem. A compact Hausdorff space (X,τ) has a unique compatible (separated) proximity.

Proof. Since a compact Hausdorff space is a normal T_1-space, we know by Theorem (2.11), that the proximity δ^* defined by (2.12) is compatible. We now show that any other compatible proximity δ on X is equivalent to δ^*, and hence conclude that the given space has a unique compatible proximity. If $A \not\delta^* B$, then $\bar{A} \cap \bar{B} = \phi$ by (2.12). But since closed and compact subsets of X coincide, Lemma (3.6) implies that $\bar{A} \not\delta \bar{B}$. Since δ is compatible, this is equivalent to $A \not\delta B$. Conversely, $A \not\delta B$ implies $\bar{A} \not\delta \bar{B}$. Thus $\bar{A} \cap \bar{B} = \phi$ which, by definition, means that $A \not\delta^* B$.

(3.8) Remarks. Contrary to Alexandroff's conjecture, there are also non-compact completely regular spaces with unique compatible proximities. (Such spaces are necessarily normal as will be seen in Section 7.) The

following is one such example: the space of all ordinals less than the first uncountable ordinal with the order topology is not compact, but has a unique compatible proximity.

In a later section (Theorem (7.20)), we shall actually characterize those topological spaces which possess unique compatible proximities.

(3.9) Theorem. Given a proximity space (X,δ), the relation $\ll$ has the following properties:

(i) $X \ll X$.

(ii) $A \ll B$ implies $A \subset B$. The converse holds iff (X,δ) is discrete.

(iii) $A \subset B \ll C \subset D$ implies $A \ll D$.

(iv) $A \ll B_i$ for $i = 1,\ldots,n$ iff $A \ll \bigcap_{i=1}^{n} B_i$.

(v) $A \ll B$ implies $(X - B) \ll (X - A)$.

(vi) $A \ll B$ implies there is a C such that $A \ll C \ll B$.

For a separated proximity,

(vii) $x \ll (X - y)$ iff $x \neq y$.

Proof: (i) Since $X \not\delta \emptyset$ by (1.3), $X \ll X$.

(ii) If $A \not\delta (X - B)$ then $A \cap (X - B) = \emptyset$, implying $A \subset B$. The second part is an easy consequence of the definition of discrete proximity given in Example (1.9).

(iii) If $A \not\ll D$ then $A \,\delta\, (X - D)$. This implies $B \,\delta\, (X - C)$ or $B \not\ll C$, a contradiction.

(iv) It suffices to consider $n = 2$. $A \ll B_1$ and $A \ll B_2$ iff $A \not\delta (X - B_1)$ and $A \not\delta (X - B_2)$ iff (by (1.2)) $A \not\delta [(X - B_1) \cup (X - B_2)]$ iff $A \not\delta [X - (B_1 \cap B_2)]$ iff $A \ll (B_1 \cap B_2)$.

(v) $A \ll B$ implies $A \not\delta (X - B)$. By (1.1), $(X - B) \not\delta A$, i.e. $(X - B) \ll (X - A)$.

(vi) $A \ll B$ implies $A \not\delta (X - B)$. By the Strong Axiom, there exists a set $(X - C)$ such that $A \not\delta (X - C)$ and $C \not\delta (X - B)$; that is, $A \ll C \ll B$.

(vii) $x \neq y$ iff $x \not\delta y$ (by (1.6)) iff $x \ll (X - y)$.

(3.10) Corollary. $A_i \ll B_i$ for $i = 1,\ldots,n$ implies $\bigcap_{i=1}^{n} A_i \ll \bigcap_{i=1}^{n} B_i$ and $\bigcup_{i=1}^{n} A_i \ll \bigcup_{i=1}^{n} B_i$.

All of the separated proximity axioms (1.1)-(1.6) are used in the above proofs. In particular we note that (1.4) is equivalent to 3.9 (vi), and (1.6) is equivalent to 3.9 (vii). The following is a converse of Theorem (3.9).

(3.11) Theorem. If $\ll$ is a binary relation on 2^X satisfying 3.9 (i)-(vi) and δ is defined by:

(3.12) $$A \not\delta B \text{ iff } A \ll (X - B) ,$$

then δ is a proximity on X . B is δ-neighbourhood of A iff $A \ll B$.

Moreover, if $\ll$ also satisfies 3.9 (vii), then δ is separated.

Proof: (i) $A \not\delta B$ implies $A \ll (X - B)$. By 3.9 (v), $B \ll (X - A)$, and so $B \not\delta A$.

(ii) $(A \cup B) \not\delta C$ implies $(A \cup B) \ll (X - C)$. Then by 3.9 (iii), $A \ll (X - C)$ and $B \ll (X - C)$, i.e. $A \not\delta C$ and $B \not\delta C$. Conversely, if $(A \cup B)\, \delta\, C$ then by (i), $C\, \delta\, (A \cup B)$. Hence $C \not\ll [X - (A \cup B)]$, or $C \not\ll [(X - A) \cap (X - B)]$. Then by 3.9 (iv), $C \not\ll (X - A)$ or $C \not\ll (X - B)$. Hence $C\, \delta\, A$ or $C\, \delta\, B$ and it follows, since δ is symmetric, that $A\, \delta\, C$ or $B\, \delta\, C$.

(iii) (1.3) is a direct consequence of 3.9 (i).

(iv) Suppose $A \not\delta B$, i.e. $A \ll (X - B)$. Then 3.9 (vi) assures the existence of a C such that $A \ll (X - C) \ll (X - B)$. Thus there is a C such that $A \not\delta C$ and $(X - C) \not\delta B$.

(v) If $A \not\delta B$, then $A \ll (X - B)$. From 3.9 (ii) we have $A \subset (X - B)$, i.e. $A \cap B = \phi$.

If 3.9 (vii) is satisfied, (1.6) follows immediately. That B is a δ-neighbourhood of A iff $A \ll B$ follows easily from the definitions of the terms involved.

In a uniform space $(X,\underset{\sim}{U})$, the closure of any subset A is given by

$$\overline{A} = \bigcap_{U \in \underset{\sim}{U}} U[A] \quad .$$

The following result is an analogue of this in proximity spaces.

(3.13) **Theorem.** If (X,δ) is a proximity space and $A \subset X$, then

$$\overline{A} = \bigcap_{A << B} B \quad .$$

Proof: From 3.2 (i) and 3.9 (ii) we conclude that $A << B$ implies $\overline{A} \subset B$, and hence $\overline{A} \subset \bigcap_{A << B} B$. To show the reverse inclusion, suppose that $x \notin \overline{A}$. Then $x \not\delta \overline{A}$ and, by (3.5), $\overline{A}$ has a δ-neighbourhood B_x not containing x. Thus $x \notin \bigcap_{A << B} B$.

In Theorem (2.10) it was shown that every completely regular space (X,τ) has a compatible proximity δ. The following is a converse.

(3.14) **Theorem.** If (X,δ) is a (separated) proximity space, then $\tau(\delta)$ is (Tychonoff) completely regular.

Proof: That $\tau(\delta)$ is T_1 if δ is separated follows easily from Axiom (1.6). We now indicate briefly why $\tau(\delta)$ is completely regular. If A is a closed set and $x \notin A$, then $x \not\delta A$. Hence we have $x << (X - A)$ and, after applying 3.9 (vi) twice, we find that there are sets B and C such that $x << B << C << (X - A)$. Using Lemma (3.2), we obtain

$x \subset \text{Int } B \subset \overline{B} \subset \text{Int } C \subset \overline{C} \subset (X - A)$. This is similar to the main step in the proof of Urysohn's Lemma, and proceeding as therein, we obtain a continuous function $f : X \to [0,1]$ such that $f(x) = 0$ and $f(A) = 1$.

(3.15) Remarks. Actually, following the proof of Urysohn's Lemma one can prove that for a compatible proximity δ defined on X , $A \not\delta B$ implies the existence of a continuous function $f : X \to [0,1]$ such that $f(A) = 0$ and $f(B) = 1$ (see Theorem (7.12)). Consequently, the proximity defined by (1.11), namely $A \not\delta B$ iff A and B are functionally distinguishable, is the largest or finest compatible proximity which can be defined on a completely regular space. Or, as Smirnov proved: every completely regular space has a maximal associated proximity space. It is useful to note that, as a result of Corollary (2.13), the proximity defined by (2.12) is the largest compatible proximity that can be defined on a normal T_1-space.

We can now strengthen (2.17) (b) to read: Let τ_1 and τ_2 be two completely regular topologies on X , let δ_1 be any proximity on X compatible with τ_1 , and let δ_2 be defined by (1.11) with respect to τ_2 . Then $\tau_1 \leq \tau_2$ implies $\delta_1 \leq \delta_2$.

§4. Subspaces. Product and Quotient Proximity Spaces.

In the study of general topological spaces, continuous functions play an important role. A similar role is played by uniformly continuous functions in uniform spaces. Their analogue in the theory of proximity spaces is the concept of a proximity (or proximally continuous or equicontinuous or δ-) mapping.

(4.1) Definition. Let (X,δ_1) and (Y,δ_2) be two proximity spaces. A function $f : X \to Y$ is said to be a proximity mapping iff

$$A \, \delta_1 \, B \text{ implies } f(A) \, \delta_2 \, f(B) .$$

Equivalently, f is a proximity mapping iff

$$C \not\delta_2 D \text{ implies } f^{-1}(C) \not\delta_1 f^{-1}(D)$$

or

$$C \ll D \text{ implies } f^{-1}(C) \ll f^{-1}(D) .$$

It is easy to see that the composition of two proximity mappings is a proximity mapping. The following is similar to the well-known result: a uniformly continuous function is continuous with respect to the induced topologies.

(4.2) Theorem. A proximity mapping $f : (X,\delta_1) \to (Y,\delta_2)$ is continuous with respect to $\tau(\delta_1)$ and $\tau(\delta_2)$.

Proof: The result follows easily from the fact that $x \, \delta_1 \, A$ implies $f(x) \, \delta_2 \, f(A)$, i.e. $f(\overline{A}) \subset \overline{f(A)}$.

(4.3) Remarks. The converse of the above theorem is false. Consider the example of (2.18): the identity mapping on X is continuous with respect to $\tau(\delta_1)$ and $\tau(\delta_2)$, but is not a proximity mapping from (X,δ_1) to (X,δ_2).

It is natural to inquire as to when the converse of Theorem (4.2) is true. Recalling that a continuous function on a compact space is uniformly continuous, we observe from the next theorem that a completely analogous result holds in proximity spaces. However, compactness is too strong and we shall prove a better result when we consider equinormal proximity spaces (see Theorem (7.22)).

(4.4) Theorem. If (X,δ_1) and (Y,δ_2) are proximity spaces and X is compact, then every continuous function f from X to Y is a proximity mapping.

Proof. If A and B are subsets of X such that $A \delta_1 B$, then $\overline{A} \cap \overline{B} \neq \emptyset$ by Theorem (3.7). But this implies that $f(\overline{A}) \cap f(\overline{B}) \neq \emptyset$, i.e. $f(\overline{A}) \delta_2 f(\overline{B})$. Since f is continuous, $f(\overline{A}) \subset \overline{f(A)}$ and $f(\overline{B}) \subset \overline{f(B)}$, yielding $\overline{f(A)} \delta_2 \overline{f(B)}$. From Lemma (2.8) it follows that $f(A) \delta_2 f(B)$, and we conclude that f is a proximity mapping.

(4.5) Theorem. Given a function $f : X \to (Y,\delta_2)$, the coarsest proximity δ_o, which may be assigned to X in order that f is proximally continuous, is defined by

(4.6) $A \not\delta_0 B$ iff there exists a $C \subset Y$ such that $f(A) \not\delta_2 (Y - C)$ and $f^{-1}(C) \subset (X - B)$.

Proof. We first verify that δ_0 is a proximity on X .

(i) Suppose $A \not\delta_0 B$ and let $D = (Y - f(A))$. Since $f(B) \subset (Y - C)$ and $f(A) \not\delta_2 (Y - C)$, we have $f(B) \not\delta_2 (Y - D)$. Moreover, $f^{-1}(D) = X - f^{-1}(f(A)) \subset (X - A)$. Hence $B \not\delta_0 A$.

(ii) $(A \cup B) \not\delta_0 C$ implies the existence of a $D \subset Y$ such that $[f(A) \cup f(B)] \not\delta_2 (Y - D)$ and $f^{-1}(D) \subset (X - C)$, from which $A \not\delta_0 C$ and $B \not\delta_0 C$ follow. If $A \not\delta_0 C$ and $B \not\delta_0 C$, there exist D_1 and D_2 such that $f(A) \not\delta_2 (Y - D_1)$, $f(B) \not\delta_2 (Y - D_2)$, $f^{-1}(D_1) \subset (X - C)$ and $f^{-1}(D_2) \subset (X - C)$. Therefore $[f(A) \cup f(B)] \not\delta_2 [Y - (D_1 \cup D_2)]$ and $f^{-1}(D_1 \cup D_2) \subset (X - C)$, i.e. $(A \cup B) \not\delta_0 C$.

(iii) If $A = \emptyset$, then $f(A) \not\delta_2 Y$ and $f^{-1}(\emptyset) \subset (X - B)$. Hence $A \not\delta_0 B$.

(iv) If $A \not\delta_0 B$, then there exists a $C \subset Y$ such that $f^{-1}(C) \subset (X - B)$ and $f(A) \not\delta_2 (Y - C)$. This latter relation and (1.4) assure the existence of a set D such that $f(A) \not\delta_2 D$ and $(Y - D) \not\delta_2 (Y - C)$. Then $f(A) \not\delta_2 (Y - C)$, and $C \subset (Y - D)$ implies $f^{-1}(C) \subset (X - f^{-1}(D))$. On the other hand, $f^{-1}(C) \subset (X - B)$, and $f(X - f^{-1}(D)) \subset (Y - D)$ implies $f(X - f^{-1}(D)) \not\delta_2 (Y - C)$. Letting $E = f^{-1}(D)$ and using (4.6), we obtain $A \not\delta_0 E$ and $(X - E) \not\delta_0 B$.

(v) $A \not\delta_0 B$ implies there exists a $C \subset Y$ such that

$f(A) \not\delta_2 (X - C)$ and $f^{-1}(C) \subset (X - B)$. Therefore $f(A) \cap (X - C) = \phi$ and $f^{-1}(f(A)) \cap f^{-1}(X - C) = \phi$. Since $A \subset f^{-1}(f(A))$ and $B \subset f^{-1}(X - C)$, we have $A \cap B = \phi$.

In order to show that $f : (X,\delta_o) \to (Y,\delta_2)$ is proximally continuous, suppose that $f(A) \not\delta_2 f(B)$. Since $f(A) \ll (Y - f(B))$, there exists a C such that $f(A) \ll C \ll (Y - f(B))$ by 3.9 (vi). Thus $f(A) \not\delta_2 (Y - C)$ and $f^{-1}(C) \subset (X - f^{-1}(f(B)) \subset (X - B)$, i.e. $A \not\delta_o B$.

It remains to show that if δ_1 is any proximity on X such that $f : (X,\delta_1) \to (Y,\delta_2)$ is proximally continuous, then δ_1 is finer than δ_o. If $A \not\delta_o B$, then there exists a $C \subset Y$ such that $f(A) \not\delta_2 (Y - C)$ and $f^{-1}(C) \subset (X - B)$. Since f is proximally continuous, $A \not\delta_1 (X - f^{-1}(C))$, and $B \subset (X - f^{-1}(C))$ implies $A \not\delta_1 B$. Thus $\delta_1 \geq \delta_o$.

(4.7) Theorem. If $f : (X,\delta_o) \to (Y,\delta_2)$ where δ_o is defined by (4.6), then $f^{-1}[\tau(\delta_2)] \subset \tau(\delta_o)$.

Proof. Suppose $A \in f^{-1}[\tau(\delta_2)]$, so that $f(A) \in \tau(\delta_2)$. By (2.4), $x \not\delta_2 (Y - f(A))$ for each $x \in f(A)$. Since f is proximally continuous, $f^{-1}(x) \not\delta_o f^{-1}[Y - f(A)]$ implying $f^{-1}(x) \not\delta_o [X - f^{-1}(f(A))]$. Hence $f^{-1}(x) \not\delta_o (X - A)$ for each $x \in f(A)$, or $y \not\delta_o (X - A)$ for each $y \in A$. But this means $A \in \tau(\delta_o)$.

(4.8) Remarks. Theorem (4.2), stating that every proximally continuous function is continuous with respect to the induced topologies, follows

as an easy corollary to the preceding theorem. Perhaps we should also mention at this point that in Chapter III, it will be shown that every uniformly continuous function is proximally continuous with respect to the induced proximities. Moreover, a function mapping one metric space into another is uniformly continuous if and only if it is proximally continuous.

(4.9) Definition. Two proximity spaces (X,δ_1) and (Y,δ_2) are called proximally isomorphic (or δ-homeomorphic) iff there exists a one-to-one function f from X onto Y such that both f and f^{-1} are proximity mappings. Such a function f is termed a proximity (or δ-) isomorphism. The term equimorphism is also used.

It follows from Theorem (4.2) that proximally isomorphic spaces are homeomorphic. From the example of (2.18), it follows that homeomorphic spaces are not necessarily proximally isomorphic.

A property is said to be a proximity invariant iff it is preserved under proximity isomorphisms. That every topological invariant is a proximity invariant is clear, again from Theorem (4.2).

(4.10) Definition. If (X,δ) is a proximity space and $Y \subset X$, then subsets of Y are also subsets of X. For subsets A and B of Y, we define

(4.11) $$A \,\delta_Y\, B \text{ iff } A \,\delta\, B \;.$$

It is easily verified that δ_Y is a proximity on Y and that

$\tau(\delta_Y)$ is the subspace topology induced on Y by $\tau(\delta)$. We call δ_Y the induced (or subspace) proximity.

We next consider the product of a family $\{(X_a,\delta_a) : A \in \Lambda\}$ of proximity spaces. Let $X = \prod_{a \in \Lambda} X_a$ denote the Cartesian product of these spaces. A proximity δ can be defined on X as follows:

(4.12) Let A and B be subsets of X. Define

$A \delta B$ iff for each pair of finite covers $\{A_i : i = 1,\ldots,m\}$ and $\{B_j : j = 1,\ldots,n\}$ of A and B respectively, there exist an A_i and a B_j such that $P_a[A_i] \delta_a P_a[B_j]$ for each $a \in \Lambda$. (P_a denotes the projection of X onto X_a.)

(4.13) Theorem. The binary relation δ defined in (4.12) is a proximity on the product space X. It is separated iff each δ_a is separated.

Proof. (i) Since each δ_a is symmetric, so is δ and (1.1) is satisfied.

(ii) Let A, B and C be subsets of X and suppose $A \delta C$. Since every cover of $(A \cup B)$ is a cover of A, it follows that $(A \cup B) \delta C$. Conversely, suppose $A \not\delta C$ and $B \not\delta C$. Then there are finite covers $\{A_i : i = 1,\ldots,m\}$ and $\{C_j : j = 1,\ldots,n\}$ of A and C respectively such that $P_a[A_i] \not\delta_a P_a[C_j]$ for some $a = s_{ij} \in \Lambda$ where $i = 1,\ldots,m$ and $j = 1,\ldots,n$. Likewise, there are finite covers $\{A_i : i = m+1,\ldots,m+p\}$ and $\{D_k : k = 1,\ldots,q\}$ of B and

C respectively such that $P_a[A_i] \not\delta_a P_a[D_k]$ for some $a = t_{ik} \in \Lambda$ where $i = m+1,\ldots,m+p$ and $k = 1,\ldots,q$. Now $\{C_j \cap D_k : j = 1,\ldots,n; k = 1,\ldots,q\}$ is a cover of C and $\{A_i : i = 1,\ldots,m+p\}$ is a cover of $A \cup B$. Since $P_a[A_i] \not\delta_a P_a[C_j \cap D_k]$ for $a = s_{ij}$ or $a = t_{ik}$, we conclude that $(A \cup B) \not\delta C$.

(iii) That $A \,\delta\, B$ implies that A and B are non-void follows easily.

(iv) If $A \not\delta B$, then there exist covers $\{A_i : i = 1,\ldots,m\}$ and $\{B_j : j = 1,\ldots,n\}$ of A and B respectively such that $P_s[A_i] \not\delta_s P_s[B_j]$ for some $s = t_{ij} \in \Lambda$ where $i = 1,\ldots,m$ and $j = 1,\ldots,n$. Since each (X_a,δ_a) is a proximity space, there exist sets E_{ij} such that $P_s[A_i] \not\delta_s E_{ij}$ and $(X - E_{ij}) \not\delta_s B_j$. Let Q denote the inverse of P_a for each $a \in \Lambda$, and let $E_j = \bigcap_{i=1}^{n} Q[E_{ij}]$ and $E = \bigcup_{j=1}^{n} E_j$. Since $P_s[E_j] \subset E_{ij}$, we have $P_s[A_i] \not\delta_s P_s[E_j]$ for $s = t_{ij}$; that is, $A \not\delta E$. Let $D_{ij} = (X - Q[E_{ij}]) = Q[X - E_{ij}]$ and $F_j = (X - E_j) = \bigcup_{j=1}^{m} D_{ij}$. Thus $(X - E) = \bigcap_{j=1}^{n} F_j$, from which we obtain $B_j \not\delta D_{ij}$ for all i and j. This implies $B_j \not\delta F_j$ for all j; i.e. $(X - E) \not\delta B_j$ for all j, showing that $(X - E) \not\delta B$.

(v) If $A \cap B \neq \phi$, then there exists an $x = (x_a) \in A \cap B$. For every pair of covers $\{A_i : i = 1,\ldots,m\}$ and $\{B_j : j = 1,\ldots,n\}$ of A and B respectively, there exists an A_i and a B_j containing x. Clearly $x_a \in P_a[A_i] \cap P_a[B_j]$ for all $a \in \Lambda$, which implies $P_a[A_i] \,\delta_a\, P_a[B_j]$ for $a \in \Lambda$;

that is, $A \, \delta \, B$.

(vi) Suppose X is separated and $x_a \, \delta_a \, y_a$ for points x_a, y_a in X_a . There are two points in X , x and y say, such that $x = (x_\lambda)$ and $y = (y_\lambda)$ where $x_\lambda = y_\lambda$ for all $\lambda \in \Lambda$ except $\lambda = a$. Then $x \, \delta \, y$, implying $x = y$. That is $x_a = y_a$, showing δ_a is separated. Conversely, $x = (x_a) \, \delta \, y = (y_a)$ implies $x_a \, \delta_a \, y_a$ for all $a \in \Lambda$. If each δ_a is separated, then $x_a = y_a$ for all $a \in \Lambda$, i.e. $x = y$.

(4.14) Corollary. Since every set covers itself, P_a is a proximity mapping from X onto X_a for each $a \in \Lambda$. In fact, δ is the smallest proximity on X for which each projection P_a is proximally continuous.

Proof. To prove the second part, suppose that $\delta \geq \alpha$, a proximity on X such that each projection map is proximally continuous; i.e. $A \, \delta \, B$ implies $A \, \alpha \, B$. Suppose $A \, \alpha \, B$, $\{A_i : i = 1, \ldots, n\}$ covers A and $\{B_j : j = 1, \ldots, m\}$ covers B . Since α is a proximity, (1.2) implies that $A_i \, \alpha \, B_j$ for some i and j . But each projection map is proximally continuous, so that $P_a[A_i] \, \delta_a \, P_a[B_j]$ for all $a \in \Lambda$. Hence $A \, \delta \, B$ and therefore $\alpha = \delta$.

The above corollary reminds one of the manner in which the Tychonoff product topology is defined on the Cartesian product of a

collection of topological spaces. Similarly, an analogous definition to that of the quotient topology is used in the case of quotient proximity.

(4.15) Definition. Let f be a function on a proximity space (X,δ) with range Y. The quotient proximity $f^q(\delta)$ for Y is the finest proximity such that f is proximally continuous.

(4.16) Theorem. If $f : (X,\delta) \to Y$, then the topology induced by the quotient proximity δ_o for Y is smaller than the quotient topology $f^q[\tau(\delta)]$ for Y.

Proof. Suppose $A \in \tau(\delta_o)$. Then $x \not\delta_o (Y - A)$ for every $x \in A$. It suffices to show that $f^{-1}(A) \in \tau(\delta)$, i.e. $x \not\delta (X - f^{-1}(A))$ for every $x \in f^{-1}(A)$. Suppose otherwise, that there exists an $x_o \in f^{-1}(A)$ such that $x_o \delta (X - f^{-1}(A))$. Since f is proximally continuous, $f(x_o) \delta_o (Y - A)$ where $f(x_o) \in A$, yielding a contradiction to $A \in \tau(\delta_o)$. Hence $A \in f^q[\tau(\delta)]$.

(4.17) Definition. Given a function $f : (X,\delta) \to Y$, the open image is defined to be the coarsest proximity structure δ_o on Y such that $A \ll B$ implies $f(A) \ll f(B)$, i.e. $A \not\delta (X - B)$ implies $f(A) \not\delta_o (Y - f(B))$. A function $f : (X,\delta) \to (Y,\delta_1)$ is said to be proximally open iff $\delta_1 \geq \delta_o$.

(4.18) Theorem. If $f : (X,\delta) \to (Y,\delta_1)$ is proximally open, then it is

an open mapping of $(X,\tau(\delta))$ to $(Y,\tau(\delta_1))$.

Proof. If $A \in \tau(\delta)$, then $x \not\delta (X - A)$ for every $x \in A$. Then since f is proximally open, we have $f(x) \not\delta_1 (Y - f(A))$ for every $f(x) \in f(A)$, i.e. $f(A) \in \tau(\delta_1)$.

NOTES

§1. The axioms for a proximity space were originally given by Efremovič [17, 18], although they appeared in a slightly different but equivalent form to those presented in this section. A proof of Efremovič's theorem, that f is uniformly continuous iff $D(A,B) = 0$ implies $E(f(A), f(B)) = 0$, can be found on Page 70 of Mamuzič [59]. The equivalence of the Symmetry Axiom for a proximity to the complete regularity of the associated topology has been shown by Pervin [76].

§2,3. The theorems in these sections are mainly due to Efremovič [18], as is the concept of a δ-neighbourhood. They have been collectively presented by Smirnov [87,88] in his early survey of proximity spaces. In the same survey, Smirnov announced that every completely regular space has a maximal associated proximity structure. The example referred to in Remarks (3.8) was contributed by Pervin [78]. For a clear proof of Urysohn's lemma, the reader is referred to Page 100 of Thron [102]. It should be observed that, as with most order relations, there is no general agreement on the definition of a partial order on proximities.

§4. The results concerning proximity mappings were first established by Smirnov [88]. A brief discussion of most of the topics covered in this section can be found in Dowker [16]. For an account of proximity on the product of proximity spaces, see Leader [52].

CHAPTER II

COMPACTIFICATIONS OF PROXIMITY SPACES

§5. Clusters and Ultrafilters

Ultrafilters play an important role in topological spaces inasmuch as such notions as convergence and compactness can be characterized in terms of ultrafilters. In this section we consider their counterparts in proximity spaces, namely clusters. We show that ultrafilters and clusters are closely related, and use this relationship to derive several important results in the theory of proximity spaces.

If $\underset{\sim}{L}$ is an ultrafilter in a set X, then it is well-known that the sets in $\underset{\sim}{L}$ are characterized by the following conditions:

(5.1) If A and B belong to $\underset{\sim}{L}$, then $A \cap B \neq \emptyset$.

(5.2) If $A \cap C \neq \emptyset$ for every $C \in \underset{\sim}{L}$, then $A \in \underset{\sim}{L}$.

(5.3) If $(A \cup B) \in \underset{\sim}{L}$, then $A \in \underset{\sim}{L}$ or $B \in \underset{\sim}{L}$.

It is natural to expect that the collections of sets in a proximity space satisfying conditions similar to (5.1) - (5.3) with nearness replacing non-empty intersection should be valuable in the theory of proximity spaces. This is indeed the case and we are led to the following definition:

(5.4) Definition. A collection σ of subsets of a proximity space (X,δ) is called a cluster iff the following conditions are satisfied:

(i) If A and B belong to σ, then $A \,\delta\, B$.

(ii) If $A \,\delta\, C$ for every $C \in \sigma$, then $A \in \sigma$.

(iii) If $(A \cup B) \in \sigma$, then $A \in \sigma$ or $B \in \sigma$.

(5.5) Remarks. (i) For each $x \in X$, the collection

$$\sigma_x = \{A \subset X : A \,\delta\, x\}$$

is a cluster. We call such a cluster a point cluster and use the above notation.

(ii) If $\{x\} \in \sigma$ for some $x \in X$, then $\sigma = \sigma_x$. If (X,δ) is separated, then no cluster can contain more than one point since that would contradict (1.6); consequently, $x \neq y$ implies $\sigma_x \neq \sigma_y$.

(iii) If σ is any cluster in X, then $X \in \sigma$ by 5.4 (ii). Hence for each subset E of X, either $E \in \sigma$ or $(X - E) \in \sigma$. Recall that an ultrafilter also has this property.

(iv) If $A \in \sigma$ and $A \subset B$, then $B \in \sigma$. This too is a property of an ultrafilter.

(5.6) Lemma. If σ_1 and σ_2 are two clusters in a proximity space (X,δ) such that $\sigma_1 \subset \sigma_2$, then $\sigma_1 = \sigma_2$.

Proof. If $A \in \sigma_2$ then $A \delta C$ for every $C \in \sigma_2$. Since $\sigma_1 \subset \sigma_2$, $A \delta B$ for every $B \in \sigma_1$ which shows that $A \in \sigma_1$. Thus $\sigma_2 \subset \sigma_1$.

The following lemma on ultrafilters is useful in deriving the fundamental relationship (Theorem (5.8)) between ultrafilters and clusters, and the two together are repeatedly used in many proofs.

(5.7) Lemma. Let $\underset{\sim}{P}$ be a collection of subsets of X such that (i) $\phi \notin \underset{\sim}{P}$, and (ii) $(A \cup B) \in \underset{\sim}{P}$ iff $A \in \underset{\sim}{P}$ or $B \in \underset{\sim}{P}$. If $A_o \in \underset{\sim}{P}$, then there exists an ultrafilter $\underset{\sim}{L}$ such that

(a) $A_o \in \underset{\sim}{L}$

(b) $A_i \in \underset{\sim}{L}$ for $i = 1, \ldots, n$ implies $\bigcap_{i=1}^{n} A_i \in \underset{\sim}{P}$.

Proof. By Zorn's lemma there exists a maximal collection $\underset{\sim}{L}$ (of subsets of X) satisfying (a) and (b). Obviously $\phi \notin \underset{\sim}{L}$. If A and B belong to $\underset{\sim}{L}$ then by (b), $A \cap B \in \underset{\sim}{P}$. Since $\underset{\sim}{L}$ is maximal, we must have $A \cap B \in \underset{\sim}{L}$. If $A \in \underset{\sim}{L}$ and $A \subset D$, then $D \in \underset{\sim}{P}$ and hence belongs to $\underset{\sim}{L}$ since $\underset{\sim}{L}$ is maximal. Having shown that $\underset{\sim}{L}$ is a filter, it remains to show that $\underset{\sim}{L}$ is an ultrafilter. Supposing the contrary, there would exist a subset E of X such that neither E nor $(X - E)$ belongs to $\underset{\sim}{L}$. Hence there are sets A_1 and A_2 in $\underset{\sim}{L}$ such that neither $A_1 \cap E$ nor $A_2 \cap (X - E)$ belongs to $\underset{\sim}{P}$. If $A = A_1 \cap A_2$, then $A \in \underset{\sim}{P}$ but neither $A \cap E$ nor $A \cap (X - E)$ belongs to $\underset{\sim}{P}$, a contradiction.

(5.8) **Theorem.** A collection σ of subsets of a proximity space (X,δ) is a cluster if and only if there exists an ultrafilter $\underset{\sim}{L}$ in X such that:

(5.9) $$\sigma = \{A \subset X : A \,\delta\, B \text{ for every } B \in \underset{\sim}{L}\} .$$

Moreover, given σ and $A_o \in \sigma$, there exists an $\underset{\sim}{L}$ satisfying (5.9) which contains A_o.

Proof: Let $\underset{\sim}{L}$ be an ultrafilter in X and let σ be defined by (5.9). We shall first show that σ is a cluster.

(i) Suppose A and B belong to σ. For every subset C, either C or $(X - C)$ is in $\underset{\sim}{L}$. This means that both A and B are near to either C or $(X - C)$. Hence for every subset C, either $A \,\delta\, C$ or $(X - C) \,\delta\, B$ which shows (by the Strong Axiom) that $A \,\delta\, B$.

(ii) Since $\underset{\sim}{L} \subset \sigma$, 5.4 (ii) follows easily.

(iii) If neither A nor B belongs to σ, there exist A' and B' in $\underset{\sim}{L}$ such that $A \not\delta A'$ and $B \not\delta B'$. Using (1.2), we obtain $A \not\delta (A' \cap B')$ and $B \not\delta (A' \cap B')$. Thus $(A \cup B) \not\delta (A' \cap B')$. Since $(A' \cap B') \in \underset{\sim}{L}$, it follows that $(A \cup B) \notin \sigma$ as required.

Conversely, let σ be a cluster and $A_o \in \sigma$. Taking $\underset{\sim}{P} = \sigma$ in Lemma (5.7), we obtain an ultrafilter $\underset{\sim}{L}$ such that (a) $A_o \in \underset{\sim}{L}$, and (b) $A_i \in \underset{\sim}{L}$ for $i = 1,\dots,n$ implies

$\bigcap_{i=1}^{n} A_i \in \sigma$. Clearly $\underset{\sim}{L} \subset \sigma$. If $\sigma' = \{A \subset X : A \ \delta \ B$ for every $B \in \underset{\sim}{L}\}$, then $\sigma \subset \sigma'$ and by Lemma (5.6), $\sigma = \sigma'$.

(5.10) Corollary. If an ultrafilter $\underset{\sim}{L} \subset \sigma$, then σ is uniquely determined.

(5.11) Remarks. (i) If $\underset{\sim}{L}$ and σ are as in Theorem (5.8), we say that " $\underset{\sim}{L}$ generates σ " or " σ is determined by $\underset{\sim}{L}$ " .

(ii) In Theorem (5.8), $\underset{\sim}{L}$ need only be an ultrafilter base.

(iii) A given cluster σ may be determined by different ultrafilters, as the following example shows:

Let X be the real line with the usual topology and let δ be defined by $A \ \delta \ B$ iff $\overline{A} \cap \overline{B} \neq \emptyset$. Then δ is a compatible proximity by Theorem (2.11). Now consider the point cluster σ_0 . Let $\underset{\sim}{L}_1$ and $\underset{\sim}{L}_2$ be ultrafilters containing the filter bases $\underset{\sim}{F}_1 = \{(-a,0) : a > 0\}$ and $\underset{\sim}{F}_2 = \{(0,a) : a > 0\}$ respectively. Then $\underset{\sim}{L}_1$ and $\underset{\sim}{L}_2$ both generate σ_0, but $\underset{\sim}{L}_1 \neq \underset{\sim}{L}_2$.

(5.12) Lemma. If a cluster σ in a proximity space (X,δ) is determined by an ultrafilter $\underset{\sim}{L}$, then σ is a point cluster σ_x if and only if $\underset{\sim}{L}$ converges to x .

Proof. $\sigma = \sigma_x$ iff $\{x\} \in \sigma$ iff $x \ \delta \ A$ for every $A \in \underset{\sim}{L}$. This shows that x is a cluster point of $\underset{\sim}{L}$. But an ultrafilter converges to each of its cluster points, so that $\underset{\sim}{L}$ converges to x .

A topological space is compact if and only if every ultrafilter in the space converges to a point. The following analogue of this result follows directly from the above lemma.

(5.13) Theorem. A proximity space is compact if and only if every cluster in the space is a point cluster.

If $A \cap B \neq \phi$, then there exists an ultrafilter $\underset{\sim}{L}$ which contains both A and B. A similar result holds for clusters in proxmity spaces.

(5.14) Theorem. If $A \,\delta\, B$, then there exists a cluster σ in (X,δ) such that A and B both belong to σ.

Proof. Let $\underset{\sim}{P} = \{C \subset X : C \,\delta\, B\}$. From Lemma (5.7), there exists an ultrafilter $\underset{\sim}{L}$ containing A such that finite intersections of members of $\underset{\sim}{L}$ are near B. The cluster σ determined by $\underset{\sim}{L}$ contains both A and B.

(5.15) Remarks. (i) Theorems (5.13) and (5.14) together yield the result: two subsets of a compact proximity space are close if and only if their closures intersect. Therefore, a compact completely regular space has a unique compatible proximity (cf. Theorem (3.7)).

(ii) If $A \,\delta\, B$, then there may be different clusters σ and σ' such that A and B are members of both. For example, let X be the real line with the usual topology and define $A \,\delta\, B$ iff $\bar{A} \cap \bar{B} \neq \phi$. If $A = [0,1]$ and $B = (0,1)$, then $A \,\delta\, B$ and

both sets belong to the point clusters σ_0 and σ_1 .

We now prove a result similar to: if $\underset{\sim}{L}$ is an ultrafilter in Y and $X \subset Y$, then the trace of $\underset{\sim}{L}$ on X is an ultrafilter in X iff $X \in \underset{\sim}{L}$.

(5.16) Theorem. Let σ be a cluster in a proximity space (Y,δ) and let $X \in \sigma$. Then there exists a unique cluster σ' in (X,δ_X) such that

(i) $\sigma' \subset \sigma$

(ii) $\sigma' = \{A \subset X : A \in \sigma\}$.

Proof. By Theorem (5.8) , σ is determined by an ultrafilter $\underset{\sim}{L}$ containing X . Then $\underset{\sim}{L}_X = \{L \cap X : L \in \underset{\sim}{L}\}$, the trace of $\underset{\sim}{L}$ on X , is an ultrafilter in X and generates a cluster σ' in X . Since (i) follows from (ii), it suffices to prove (ii). If $A \in \sigma'$, then $A \, \delta \, (L \cap X)$ for each $L \in \underset{\sim}{L}$. This implies $A \, \delta \, L$ for each $L \in \underset{\sim}{L}$, i.e. $A \in \sigma$, and hence $\sigma' \subset \{A \subset X : A \in \sigma\}$. To prove the reverse inclusion, suppose $A \subset X$ and $A \in \sigma$. Then $A \, \delta \, (L \cap X)$ for each $L \in \underset{\sim}{L}$, i.e. $A \in \sigma'$. Uniqueness is shown by a method similar to that used in Lemma (5.6).

The following result corresponds to the well-known theorem: if $\underset{\sim}{L}$ is an ultrafilter in X and $X \subset Y$, then $\underset{\sim}{L}$ is an ultrafilter base in Y and thus generates an ultrafilter in Y .

(5.17) Theorem. If X is a subspace of a proximity space (Y,δ), then every cluster σ' in X is a subclass of a unique cluster σ in Y, and

$$\sigma = \{A \subset Y : A \, \delta \, B \text{ for every } B \in \sigma'\} .$$

Proof. σ' is determined by an ultrafilter $\underset{\sim}{L}$ in X which is an ultrafilter base in Y. From 5.11 (ii), it follows that $\underset{\sim}{L}$ generates a cluster σ in Y. Moreover, $A \in \sigma$ iff $A \, \delta \, B$ for each $B \in \underset{\sim}{L}$ iff $A \, \delta \, C$ for each $C \in \sigma'$. Uniqueness follows as usual.

Let f be a function from X onto Y. The following results are well-known:

(a) If $\underset{\sim}{L}$ is an ultrafilter in X, then $f(\underset{\sim}{L}) = \{f(L) : L \in \underset{\sim}{L}\}$ is an ultrafilter in Y.

(b) If X and Y are topological spaces, $\underset{\sim}{L}$ is an O-filter in X and f is continuous, then $f(\underset{\sim}{L})$ is an O-filter in Y. (Recall that $\underset{\sim}{L}$ is an O-filter in (X,τ) iff for every $G \in \tau$, either $G \in \underset{\sim}{L}$ or $(X - G) \in \underset{\sim}{L}$.)

We find the following analogue in proximity spaces.

(5.18) Theorem. If f is a proximity mapping from (X,δ_1) to (Y,δ_2) and σ_1 is a cluster in X, then there corresponds a cluster σ_2 in Y such that:

$$\sigma_2 = \{A \subset Y : A\ \delta_2\ f(B) \text{ for each } B \in \sigma_1\} .$$

Proof. σ_1 is determined by an ultrafilter $\underset{\sim}{L}$ in X. Then $f(\underset{\sim}{L})$ is an ultrafilter in Y and generates a cluster σ_2 in Y. If $A\ \delta_2\ f(B)$ for every $B \in \sigma_1$, then $A\ \delta_2\ f(L)$ for every $L \in \underset{\sim}{L}$ and hence $A \in \sigma_2$. To prove the reverse inclusion, we first note that $f(\sigma_1) \subset \sigma_2$. This follows from the fact that if $B \in \sigma_1$, then $B\ \delta_1\ L$ for every $L \in \underset{\sim}{L}$, and f being a proximity mapping implies $f(B)\ \delta_2\ f(L)$ for each $L \in \underset{\sim}{L}$, i.e. $f(B) \in \sigma_2$. Thus if $A \in \sigma_2$, then $A\ \delta_2\ f(B)$ for every $B \in \sigma_1$.

We now present an axiomatic characterization of the family of all clusters in a separated proximity space.

(5.19) Definition. A semi-ultrafilter $\underset{\sim}{S}$ in a set X is a collection of non-empty subsets of X satisfying the following conditions:

(i) If $(A \cup B) \in \underset{\sim}{S}$, then $A \in \underset{\sim}{S}$ or $B \in \underset{\sim}{S}$.

(ii) If $A \in \underset{\sim}{S}$ and $A \subset B$, then $B \in \underset{\sim}{S}$.

(iii) If $A \cap S \neq \emptyset$ for every $S \in \underset{\sim}{S}$, then $A \in \underset{\sim}{S}$.

(iv) If $\{x\}$ and $\{y\}$ both belong to $\underset{\sim}{S}$, then $x = y$.

Although every ultrafilter is clearly a semi-ultrafilter, the converse is not true. Every cluster in a separated proximity space is a semi-ultrafilter, showing that a semi-ultrafilter is not necessarily a filter.

(5.20) **Theorem.** Let $\underset{\sim}{C}$ denote the family of all clusters in a separated proximity space (X,δ). Then the following conditions are satisfied:

(i) Let A and B be subsets of X. If for every $E \subset X$ there is a $\sigma \in \underset{\sim}{C}$ such that either $A, E \in \sigma$ or $(X - E), B \in \sigma$, then there is a $\sigma' \in \underset{\sim}{C}$ such that $A, B \in \sigma'$.

(ii) Consider any $\sigma \in \underset{\sim}{C}$. If for each $B \in \sigma$ there is a $\sigma' \in \underset{\sim}{C}$ such that $A, B \in \sigma'$, then $A \in \sigma$.

(iii) Each ultrafilter in X is contained in some member of $\underset{\sim}{C}$.

Proof. (i) is a consequence of the Strong Axiom and Theorem (5.14). Condition (ii) follows from the fact that any two sets in a cluster are near to one another, and (iii) is proved in Theorem (5.8).

(5.21) **Theorem.** If $\underset{\sim}{C}$ is a family of semi-ultrafilters in X satisfying conditions 5.20 (i), (ii) and (iii), then there exists a separated proximity δ on X such that $\underset{\sim}{C}$ is the family of all clusters in (X,δ).

Proof. If $\underset{\sim}{C}$ is a family of semi-ultrafilters satisfying 5.20 (i), (ii) and (iii), define δ by:

(5.22) $A \,\delta\, B$ iff there is a $\sigma \in \underset{\sim}{C}$ such that $A, B \in \sigma$.

(Theorem (5.14) provides a motivation for this definition.)

That δ is a proximity on X will now be verified.

(1) Symmetry is obvious from (5.22).

(2) $(A \cup B)\ \delta\ C$ iff there exists a $\sigma \in \underset{\sim}{C}$ such that $(A \cup B)$, $C \in \sigma$. This is equivalent to $A, C \in \sigma$ or $B, C \in \sigma$, i.e. $A\ \delta\ C$ or $B\ \delta\ C$.

(3) That $A\ \not\delta\ \phi$ follows from the fact that ϕ does not belong to any semi-ultrafilter.

(4) If for every subset E of X either $A\ \delta\ E$ or $(X - E)\ \delta\ B$, then there exists a $\sigma \in \underset{\sim}{C}$ such that $A, E \in \sigma$ or $(X - E), B \in \sigma$. Since $\underset{\sim}{C}$ satisfies 5.20 (i) by hypothesis, there exists a $\sigma' \in \underset{\sim}{C}$ such that $A, B \in \sigma'$, i.e. $A\ \delta\ B$.

(5) $A \cap B \neq \phi$ implies A and B both belong to some ultrafilter $\underset{\sim}{L}$. Hence by 5.20 (iii), there exists a $\sigma \in \underset{\sim}{C}$ such that $\underset{\sim}{L} \subset \sigma$; that is, $A, B \in \sigma$ which implies $A\ \delta\ B$.

(6) If $x\ \delta\ y$, then x and y both belong to some $\sigma \in \underset{\sim}{C}$. Then by 5.19 (iv), $x = y$.

The next step is to prove that each $\sigma \in \underset{\sim}{C}$ is a cluster in (X, δ).

(a) If $A, B \in \sigma$ then by (5.22), $A\ \delta\ B$.

(b) If $A\ \delta\ B$ for every $B \in \sigma$, then for each $B \in \sigma$ there exists a $\sigma' \in \underset{\sim}{C}$ such that $A, B \in \sigma'$. By 5.20 (ii), $A \in \sigma$.

(c) By 5.19 (i), $(A \cup B) \in \sigma$ implies $A \in \sigma$ or $B \in \sigma$.

It now remains to show that every cluster σ in (X,δ) belongs to $\underset{\sim}{C}$. From Theorem (5.8), we know that σ is determined by some ultrafilter $\underset{\sim}{L}$ in X. By 5.20 (iii), there is a $\sigma' \in \underset{\sim}{C}$ such that $\underset{\sim}{L} \subset \sigma'$. But σ' is a cluster and, since $\underset{\sim}{L}$ generates a unique cluster, $\sigma = \sigma'$. Therefore, $\sigma \in \underset{\sim}{C}$.

(5.23) Remarks. Theorems (5.20) and (5.21) together indicate that the concept of a cluster may be considered as a primitive concept in proximity theory. We may then define a proximity space to be a set X together with a family of semi-ultrafilters satisfying 5.20 (i), (ii) and (iii). Each member of this family will be called a cluster. This leads to an alternate treatment of proximity spaces and enables one to prove the compactification of a proximity space (see Section 7) without the use of the axiom of choice.

§6. Duality in Proximity Spaces.

In Section 3, we saw an alternate way of studying proximity spaces using δ-neighbourhoods. If δ is a proximity on X, then δ and $<<$ are dual relations on 2^X in the following sense:

(6.1) **Definition.** Two relations β and β^* on 2^X are dual iff $A \beta^* B$ is equivalent to $A \not\beta (X - B)$. Clearly $\beta^{**} = \beta$, justifying the term dual.

In this section, a study is made of this duality with special reference to clusters and their duals, "ends".

(6.2) **Definition.** Let $\underset{\sim}{G}$ be a class of subsets of a proximity space (X,δ). Define:

(a) $\underset{\sim}{G}^o = \{E \subset X : \text{there exists an } A \in \underset{\sim}{G} \text{ such that } A << E\}$.

(b) $\underset{\sim}{G}' = \{E \subset X : E \,\delta\, A \text{ for every } A \in \underset{\sim}{G}\}$.

(c) $\underset{\sim}{G}^* = \{E \subset X : (X - E) \notin \underset{\sim}{G}\}$, the dual of $\underset{\sim}{G}$.

(6.3) **Lemma.** $\underset{\sim}{G}^o$ and $\underset{\sim}{G}'$ are dual classes.

Proof. $E \in \underset{\sim}{G}^o$ iff there exists an $A \in \underset{\sim}{G}$ such that $A << E$. This is equivalent to the existence of an $A \in \underset{\sim}{G}$ such that $(X - E) \not\delta A$, i.e. $(X - E) \notin \underset{\sim}{G}'$.

(6.4) **Definition.** An end $\underset{\sim}{E}$ in a proximity space (X,δ) is a collection

of subsets of X satisfying the following conditions:

(i) $B, C \in \underset{\sim}{F}$ implies there exists a non-empty $A \in \underset{\sim}{F}$ such that $A \ll B$ and $A \ll C$.

(ii) If $A \ll B$, then either $(X - A) \in \underset{\sim}{F}$ or $B \in \underset{\sim}{F}$.

(6.5) *Definition*. A *round filter* $\underset{\sim}{F}$ is a filter with the additional property that for each $F_1 \in \underset{\sim}{F}$, there exists an $F_2 \in \underset{\sim}{F}$ such that $F_2 \ll F_1$.

(6.6) *Remarks*. Clearly if $\underset{\sim}{F}$ is an end in X, then $\phi \notin \underset{\sim}{F}$ and $X \in \underset{\sim}{F}$. An example of an end, and also of a round filter, is the system $\underset{\sim}{N}_x$ of δ-neighbourhoods of a point $x \in X$. If $\underset{\sim}{F}$ is a maximal round filter converging to x, then $\underset{\sim}{F} = \underset{\sim}{N}_x$.

Some authors use the term *regular filter* rather than round filter. $\underset{\sim}{F}$ is said to be a *centred δ-system* iff (a) $A, B \in \underset{\sim}{F}$ implies $A \cap B \neq \phi$, and (b) $A \in \underset{\sim}{F}$ implies the existence of a $B \in \underset{\sim}{F}$ such that $B \ll A$. Alexandroff originally defined an end to be a maximal centred δ-system. Theorem (6.9) shows the equivalence of this definition to the one stated in (6.4), noting that a maximal round filter is the same as a maximal centred δ-system.

(6.7) *Theorem*. Every end is a maximal round filter.

Proof. Suppose that $\underset{\sim}{F}$ is an end, and begin by showing that $\underset{\sim}{F}$ is a filter. Condition 6.4 (i) and the fact that

$X \in \underset{\sim}{F}$ together show that $\underset{\sim}{F}$ is a non-empty filter base. Given that $C \in \underset{\sim}{F}$ and $C \subset D$, we must show that $D \in \underset{\sim}{F}$. By 6.4 (i), there is an $A \in \underset{\sim}{F}$ such that $A << C$. From 3.9 (iii) we have $A << D$, and condition 6.4 (ii) demands that either $(X - A) \in \underset{\sim}{F}$ or $D \in \underset{\sim}{F}$. But 6.4 (i) excludes the first possibility since $A \in \underset{\sim}{F}$, so that $D \in \underset{\sim}{F}$.

That $\underset{\sim}{F}$ is a round filter follows immediately from 6.4 (i).

Finally, we must show that the round filter $\underset{\sim}{F}$ is maximal. Let $\underset{\sim}{G}$ be a round filter such that $\underset{\sim}{F} \subset \underset{\sim}{G}$. Suppose $B \in \underset{\sim}{G}$. Then by 6.4 (i), there is an $A \in \underset{\sim}{G}$ such that $A << B$. Since $\underset{\sim}{G}$ is a filter and $A \in \underset{\sim}{G}$, we know $(X - A) \notin \underset{\sim}{F}$. Hence $B \in \underset{\sim}{F}$ by 6.4 (ii), showing that $\underset{\sim}{F} = \underset{\sim}{G}$.

(6.8) Lemma. Let $\underset{\sim}{F}$ be a round filter and $A << B$. If A intersects every member of $\underset{\sim}{F}$, then B belongs to some round filter finer than $\underset{\sim}{F}$.

Proof. Let $\underset{\sim}{G} = \{A \cap F : F \in \underset{\sim}{F}\}$ and consider $\underset{\sim}{G}^{o}$. We shall show that $\underset{\sim}{G}^{o}$ is a round filter finer than $\underset{\sim}{F}$ and that it contains B. Let P and Q be members of $\underset{\sim}{G}^{o}$. Then, by definition, there exist members C and D of $\underset{\sim}{F}$ such that $(A \cap C) << P$ and $(A \cap D) << Q$. Since $\underset{\sim}{F}$ is a filter, there exists an $E \in \underset{\sim}{F}$ such that $E \subset (C \cap D)$. From 3.9 (iii) and 3.9 (iv), it is evident that $(A \cap E) << P$, $(A \cap E) << Q$, and hence $(A \cap E) << (P \cap Q)$. Since

$(A \cap E) \in \underset{\sim}{G}$, it follows that $(P \cap Q) \in \underset{\sim}{G}^{o}$. Hence $\underset{\sim}{G}^{o}$ is a filter, since supersets of members of $\underset{\sim}{G}^{o}$ clearly belong to $\underset{\sim}{G}^{o}$. Now by 3.9 (vi), there is an R such that $(A \cap E) << R << (P \cap Q)$. By taking $P = Q$ and noting that $(A \cap E) \in \underset{\sim}{G}$ implies $R \in \underset{\sim}{G}^{o}$, we see that the filter $\underset{\sim}{G}^{o}$ is round.

To show that $\underset{\sim}{G}^{o}$ is finer than $\underset{\sim}{F}$, suppose $E \in \underset{\sim}{F}$. Since $E << X$ and $A << B$, we have $(A \cap E) << B$ by 3.9 (iv), showing that $B \in \underset{\sim}{G}^{o}$. Now $\underset{\sim}{F}$ is a round filter, implying that there exists an $F \in \underset{\sim}{F}$ such that $F << E$. By 3.9 (iii), $(A \cap F) << E$ and so $E \in \underset{\sim}{G}^{o}$.

(6.9) Theorem. $\underset{\sim}{F}$ is an end if and only if $\underset{\sim}{F}$ is a maximal round filter.

Proof. In view of Theorem (6.7), it is sufficient to show that every maximal round filter $\underset{\sim}{F}$ is an end. Condition 6.4 (i) is clearly satisfied by any round filter. In verifying 6.4 (ii), suppose $A << B$ and $B \notin \underset{\sim}{F}$. Since $\underset{\sim}{F}$ is maximal, Lemma (6.8) guarantees the existence of an $E \in \underset{\sim}{F}$ such that $A \cap E = \emptyset$. Thus $E \subset (X - A)$ and $(X - A) \in \underset{\sim}{F}$ since $\underset{\sim}{F}$ is a filter, proving 6.4 (ii).

(6.10) Corollary. Every round filter is a subclass of some end.

The following is the main result of this section, pointing out

the duality between clusters and ends.

(6.11) Theorem. $\underset{\sim}{F}$ is an end if and only if $\underset{\sim}{F}^*$ is a cluster.

Proof. If $\underset{\sim}{F}^*$ is a cluster, then:

(a^*) $A, B \in \underset{\sim}{F}^*$ implies $A \,\delta\, B$.

(b^*) $A \,\delta\, B$ for every $B \in \underset{\sim}{F}^*$ implies $A \in \underset{\sim}{F}^*$.

(c^*) $(A \cup B) \in \underset{\sim}{F}^*$ implies $A \in \underset{\sim}{F}^*$ or $B \in \underset{\sim}{F}^*$.

The dual $\underset{\sim}{F}$ of $\underset{\sim}{F}^*$ satisfies the following conditions:

(a) $A << B$ implies $(X - A) \in \underset{\sim}{F}$ or $B \in \underset{\sim}{F}$.

(b) Given $B \in \underset{\sim}{F}$, there is an E such that $(X - E) \notin \underset{\sim}{F}$ and $E << B$.

(c) $A, B \in \underset{\sim}{F}$ implies $A \cap B \in \underset{\sim}{F}$.

Note that (a) is simply a restatement of 6.4 (ii).

If $\underset{\sim}{F}$ is an end, it clearly satisfies (a) and (c). That (b) is satisfied follows from 6.4 (i), which states that given $B \in \underset{\sim}{F}$, there exists an $E \in \underset{\sim}{F}$ such that $E << B$. Since $\underset{\sim}{F}$ is a round filter by (6.7), $(X - E) \notin \underset{\sim}{F}$.

Conversely, we must show that if $\underset{\sim}{F}$ satisfies (a), (b) and (c), then $\underset{\sim}{F}$ is an end. In doing so, we need only verify 6.4 (i). In view of (c) and 3.9 (iii), it is sufficient to show that given a $B \in \underset{\sim}{F}$, there exists a non-empty $A \in \underset{\sim}{F}$

such that $A \ll B$. If $B \in \underset{\sim}{F}$, then (b) assures the existence of an E such that $E \ll B$ and $(X - E) \notin \underset{\sim}{F}$. By 3.9 (vi), there exists an A such that $E \ll A \ll B$. Since $E \ll A$ and $(X - E) \notin \underset{\sim}{F}$, we see from (a) that $A \in \underset{\sim}{F}$. Finally, $(X - E) \notin \underset{\sim}{F}$ implies $E \neq \phi$, and so $A \neq \phi$.

(6.12) Corollary. For discrete proximity spaces, ends and clusters both coincide with ultrafilters.

(6.13) Corollary. $\underset{\sim}{G}^{o}$ is an end if and only if $\underset{\sim}{G}'$ is a cluster.

(6.14) Theorem. Every ultrafilter $\underset{\sim}{F}$ in a proximity space contains a unique end $\underset{\sim}{F}^{o}$.

Proof. By Corollary (5.10), $\underset{\sim}{F}$ is a subclass of the unique cluster $\underset{\sim}{F}'$. Taking duals, we obtain $\underset{\sim}{F}'^{*} \subset \underset{\sim}{F}^{*}$. Using Lemma (6.3) and the self-duality of $\underset{\sim}{F}$, this reduces to the statement that $\underset{\sim}{F}^{o} \subset \underset{\sim}{F}$. That $\underset{\sim}{F}^{o}$ is an end follows from Corollary (6.13).

To prove uniqueness, suppose that $\underset{\sim}{G}$ is any end contained in $\underset{\sim}{F}$. Then $\underset{\sim}{G}^{o} \subset \underset{\sim}{F}^{o}$. But $\underset{\sim}{G}^{o} = \underset{\sim}{G}$ since $\underset{\sim}{G}$ is an end, and hence a maximal family. Therefore $\underset{\sim}{G} \subset \underset{\sim}{F}^{o}$, from which we deduce that $\underset{\sim}{F}^{o} = \underset{\sim}{G}$, again because an end is a maximal family.

The following is the dual of Theorem (5.14).

(6.15) Theorem. $A \ll B$ in a proximity space (X, δ) if and only if every end in X contains either $(X - A)$ or B .

Proof. Necessity is given by 6.4 (ii). To prove the converse, let $A \not\ll B$, i.e. $A \, \delta \, (X - B)$. By Theorem (5.14), there exists a cluster $\underset{\sim}{F}^*$ containing both A and $(X - B)$. Hence both $(X - A)$ and B are not in $\underset{\sim}{F}$ which, by Theorem (6.11), is an end.

The axiomatic characterization of clusters (Theorems (5.20), (5.21)) has the following dual regarding ends. The proof is omitted, being a straightforward application of the preceding theorems.

(6.16) Theorem. Let Φ be a family of filters (called Φ-filters) in X . Then Φ is the family of all ends for some separated proximity on X if and only if the following conditions are satisfied:

(i) If $\underset{\sim}{F}$ is a Φ-filter, then $\bigcap_{F \in \underset{\sim}{F}} F$ contains at most one point.

(ii) If $\underset{\sim}{F} \in \Phi$ and $B \in \underset{\sim}{F}$, then there is an $A \in \underset{\sim}{F}$ such that every Φ-filter contains either $(X - A)$ or B .

(iii) If every Φ-filter contains at least one of the sets C or D , then there is a B such that every Φ-filter which does not contain C contains B , and every Φ-filter which does not contain D contains $(X - B)$.

(iv) Every ultrafilter in X contains some Φ-filter as a subclass.

§7. Smirnov Compactification.

It should be mentioned at the outset that throughout this section, we shall work exclusively with separated proximity spaces.

In Theorem (3.7), it was shown that every compact Hausdorff space has a unique compatible separated proximity. The problem considered in this section is to determine whether or not every separated proximity space (X,δ) can be embedded in a compact proximity space Y such that the proximity δ is the subspace proximity induced by the unique proximity on Y. This can indeed be done; in fact, there is an intimate relationship between the proximities associated with a Tychonoff space and its compactifications. A one-to-one order-preserving mapping (i.e. an order-isomorphism) exists between the two, and consequently the study of proximities can be reduced to that of compactifications. We have already seen (Remarks (2.18)) that the real line has two distinct compatible proximities. These actually correspond to the two compactifications of the real line, namely the Alexandroff one-point compactification and the two-point compactification.

Using clusters, we shall now construct the Smirnov compactification of a separated proximity space. Let (X,δ) be a separated proximity space and let $\underset{\sim}{X}$ denote the set of all clusters in X. For $A \subset X$, let

$$\underset{\sim}{\bar{A}} = \{\sigma \in \underset{\sim}{X} : A \in \sigma\} .$$

For $x \in X$, let $f(x) = \sigma_x$ (the point cluster). Then it is easy to see that:

(i) f is a one-to-one mapping, and

(ii) $f(A) \subset \overline{\underset{\sim}{A}}$.

It is to obtain property (i) that we insist on the proximity δ being separated, for we are then assured that each point $x \in X$ is a member of one and only one cluster in X .

(7.1) **Definition.** For $\underset{\sim}{P} \subset \underset{\sim}{X}$, we say that a subset A of X **absorbs** $\underset{\sim}{P}$ iff $A \in \sigma$ for every $\sigma \in \underset{\sim}{P}$, i.e. $\underset{\sim}{P} \subset \overline{\underset{\sim}{A}}$.

(7.2) **Lemma.** The binary relation δ^* on $2^{\underset{\sim}{X}}$ defined by

(7.3) $\underset{\sim}{P} \, \delta^* \, \underset{\sim}{Q}$ iff A absorbs $\underset{\sim}{P}$ and B absorbs $\underset{\sim}{Q}$ implies $A \, \delta \, B$,

is a separated proximity on $\underset{\sim}{X}$.

Proof. (i) Symmetry of δ^* follows from that of δ .

(ii) Suppose that $\underset{\sim}{Q} \, \delta^* \, \underset{\sim}{R}$, that D absorbs $(\underset{\sim}{P} \cup \underset{\sim}{Q})$, and that C absorbs $\underset{\sim}{R}$. Then D absorbs $\underset{\sim}{Q}$ and hence $D \, \delta \, C$. Thus $(\underset{\sim}{P} \cup \underset{\sim}{Q}) \, \delta^* \, \underset{\sim}{R}$. Conversely, suppose that $(\underset{\sim}{P} \cup \underset{\sim}{Q}) \, \delta^* \, \underset{\sim}{R}$ and that $\underset{\sim}{P} \not\delta^* \underset{\sim}{R}$. Let B absorb $\underset{\sim}{Q}$ and C absorb $\underset{\sim}{R}$. Then we must show that $B \, \delta \, C$. Since $\underset{\sim}{P} \not\delta^* \underset{\sim}{R}$, there are sets A and D absorbing $\underset{\sim}{P}$ and $\underset{\sim}{R}$ respectively such that $A \not\delta D$. By the Strong Axiom, there is an E such that $A \not\delta E$ and $(X - E) \not\delta D$. Because D absorbs $\underset{\sim}{R}$ and $(X - E) \not\delta D$, $(X - E)$ belongs to no cluster in $\underset{\sim}{R}$. Thus $(C - E)$ belongs to no cluster in $\underset{\sim}{R}$. But

$C = (C - E) \cup (C \cap E)$ absorbs $\underset{\sim}{R}$, implying that $C \cap E$ absorbs $\underset{\sim}{R}$. Now $(A \cup B)$ absorbs $(\underset{\sim}{P} \cup \underset{\sim}{Q})$, which shows that $(A \cup B)\ \delta\ (C \cap E)$. This contradicts the fact that $A \not\delta E$ implies $A \not\delta (C \cap E)$. Thus $B\ \delta\ C$.

(iii) That $\underset{\sim}{P}\ \delta^*\ \underset{\sim}{Q}$ implies $\underset{\sim}{P}$ and $\underset{\sim}{Q}$ are non-empty follows directly from (1.3).

(iv) If $\underset{\sim}{P} \not\delta^* \underset{\sim}{Q}$, there are sets A and B absorbing $\underset{\sim}{P}$ and $\underset{\sim}{Q}$ respectively such that $A \not\delta B$. By the Strong Axiom, there is an E such that $A \not\delta E$ and $(X - E) \not\delta B$. Since $(X - E) \not\delta B$ and B absorbs $\underset{\sim}{Q}$, $(X - E)$ belongs to no cluster in $\underset{\sim}{Q}$. Thus E absorbs $\underset{\sim}{Q}$. Now let $\underset{\sim}{R} = \underset{\sim}{\overline{E}}$. Then $\underset{\sim}{P} \not\delta^* \underset{\sim}{R}$ since A absorbs $\underset{\sim}{P}$, E absorbs $\underset{\sim}{R}$ and $A \not\delta E$. Since E belongs to no cluster in $(\underset{\sim}{X} - \underset{\sim}{R})$, $(X - E)$ absorbs $(\underset{\sim}{X} - \underset{\sim}{R})$. Then $(X - E) \not\delta B$ implies $(\underset{\sim}{X} - \underset{\sim}{R}) \not\delta^* \underset{\sim}{Q}$.

(v) Suppose that $\underset{\sim}{P} \cap \underset{\sim}{Q} \neq \phi$ and that A and B absorb $\underset{\sim}{P}$ and $\underset{\sim}{Q}$ respectively. Then both A and B absorb $\underset{\sim}{P} \cap \underset{\sim}{Q}$, so that $A\ \delta\ B$ and hence $\underset{\sim}{P}\ \delta^*\ \underset{\sim}{Q}$.

(vi) A absorbs $\{\sigma\}$ where $\sigma \in \underset{\sim}{X}$ iff $A \in \sigma$. Hence $\sigma_1\ \delta^*\ \sigma_2$ iff every set in σ_1 is in σ_2 and vice versa. Thus δ^* is a separated proximity on $\underset{\sim}{X}$.

Notation. Let τ^* be the topology induced on $\underset{\sim}{X}$ by δ^* .

(7.4) Lemma. (X,δ) is proximally isomorphic to $f(X)$ with the induced proximity of δ^* , and $f(X)$ is dense in $\underset{\sim}{X}$.

Proof. We first note that A absorbs $f(B)$ iff $B \subset \overline{A}$. Hence $\sigma \, \delta^* \, f(B)$ iff $B \in \sigma$; that is, $\underset{\sim}{\overline{B}}$ is the τ^*-closure of $f(B)$. Since X belongs to every cluster, $f(X)$ is dense in $\underset{\sim}{X}$.

Now $f(A) \, \delta^* \, f(B)$ iff $C \, \delta \, D$ whenever C and D absorb $f(A)$ and $f(B)$ respectively iff $C \, \delta \, D$ whenever $A \subset \overline{C}$ and $B \subset \overline{D}$. But this last statement is equivalent to $A \, \delta \, B$, so that $f(A) \, \delta^* \, f(B)$ iff $A \, \delta \, B$. Thus X is proximally isomorphic to $f(X)$.

(7.5) Lemma. $(\underset{\sim}{X}, \tau^*)$ is compact.

Proof. By Theorem (5.13), it is sufficient to show that an arbitrary cluster σ in $\underset{\sim}{X}$ is a point cluster. Since $f(X)$ is dense in $\underset{\sim}{X}$, $f(X) \in \sigma$. From Theorem (5.16), there is a unique cluster σ' in $f(X)$ such that $\sigma' \subset \sigma$. Since X is proximally isomorphic to $f(X)$, there is a cluster σ'' in X which corresponds to σ'. To be specific, $\sigma' = \{f(A) : A \in \sigma''\}$. Referring to the proof of (7.4), we find that $\sigma'' \, \delta^* \, f(A)$ iff $A \in \sigma''$. Applying Theorem (5.17), we obtain $\sigma'' \in \sigma$, and hence σ is a point cluster.

(7.6) Lemma. $\underset{\sim}{X}$ is unique up to δ-homeomorphism.

Proof. Suppose (X, δ) is δ-homeomorphic to a dense subset of a compact proximity space (Y, δ_1). To each cluster σ

in X, there corresponds a cluster σ' in Y by Theorem (5.18). Since Y is compact, we know from Theorem (5.13) that σ' is a point cluster. By Theorem (5.16), every point (and hence cluster) in Y yields a unique cluster (via the δ-homeomorphism of the dense subspace) in X. Thus the clusters in X are exactly those determined by points in Y, so that $\underset{\sim}{X}$ and Y are δ-homeomorphic.

Combining the above sequence of lemmas, we obtain the main result of this section.

(7.7) *Theorem.* Every separated proximity space (X,δ) is a dense subspace of a unique (up to δ-homeomorphism) compact Hausdorff space $\underset{\sim}{X}$. Since $\underset{\sim}{X}$ has a unique compatible separated proximity, subsets A and B of X are near iff their closures in $\underset{\sim}{X}$ intersect ($\underset{\sim}{X}$ is called the *Smirnov compactification of* X.)

(7.8) *Corollary.* The topology of a separated proximity space is Tychonoff.

(7.9) *Remarks.* In the statement of Theorem (7.7), we have identified X with $f(X)$ as is usually done.

The Smirnov compactification of (X,δ) can also be constructed by embedding (X,δ) in the proximity space $(\underset{\sim}{Y},\delta^{**})$, where $\underset{\sim}{Y}$ is the family of all ends in X and δ^{**} is defined in either of the two following equivalent ways:

(i) For subsets π_1 and π_2 of $\underset{\sim}{Y}$, $\pi_1 \delta^{**} \pi_2$ iff for every finite collection of sets $\{B_i : i = 1,\ldots,n\}$ such that $B_i \gg A_i$ where $\bigcup_{i=1}^{n} A_i = X$, there exist $F \in \pi_1$ and $G \in \pi_2$ such that $B_k \in (F \cap G)$ for some $1 \leq k \leq n$.

(ii) Setting $O(A) = \{\alpha : A \in \alpha, \alpha \in \underset{\sim}{Y}\}$, define $\pi_1 \not\delta^{**} \pi_2$ iff there exist sets A and B such that $A \not\delta B$ and $\pi_1 \subset O(A)$ and $\pi_2 \subset O(B)$.

It can be shown that $(\underset{\sim}{Y},\tau(\delta^{**}))$ is compact. Since the Smirnov compactification is essentially unique, the duality existing between clusters and ends is again brought to our attention.

(7.10) Theorem. Every proximity mapping g of (X,δ_1) onto (Y,δ_2) has a unique extension $\bar{g}$, which is a proximity mapping of the compactification $(\underset{\sim}{X},\delta_1^*)$ of X onto the compactification $(\underset{\sim}{Y},\delta_2^*)$ of Y.

Proof. It follows from Theorem (5.18) that if σ_1 is a cluster in X, there corresponds a cluster σ_2 in Y such that $\sigma_2 = \{P \subset Y : P \,\delta_2\, g(C)$ for every $C \in \sigma_1\}$. Let $\bar{g}(\sigma_1) = \sigma_2$. Then $\bar{g}$ is a function from $\underset{\sim}{X}$ to $\underset{\sim}{Y}$. Clearly $\bar{g}$ maps the point cluster of x into the point cluster of $g(x)$. In other words, $\bar{g}$ agrees with g on X (identifying X with $f(X)$, where $f(x) = \sigma_x$). To show that $\bar{g}$ is a proximity mapping, we must show that $\underset{\sim}{P} \,\delta_1^*\, \underset{\sim}{Q}$ implies $\bar{g}(\underset{\sim}{P}) \,\delta_2^*\, \bar{g}(\underset{\sim}{Q})$, i.e. if A absorbs $\bar{g}(\underset{\sim}{P})$ and B absorbs $\bar{g}(\underset{\sim}{Q})$ then $A \,\delta_2\, B$. If $A \not\delta_2 B$, then by Lemma (3.3) there are sets C and D such that $A \not\delta_2 (Y - C)$, $B \not\delta_2 (Y - D)$ and $C \not\delta_2 D$. Since A

absorbs $\bar{g}(\underset{\sim}{P})$, $(Y - C)$ belongs to no cluster in $\bar{g}(\underset{\sim}{P})$. But g is a proximity mapping, so that $g^{-1}(Y - C) = (X - g^{-1}(C))$ belongs to no cluster in $\underset{\sim}{P}$. This shows that $g^{-1}(C)$ absorbs $\underset{\sim}{P}$. Similarly, $g^{-1}(D)$ absorbs $\underset{\sim}{Q}$. Since $\underset{\sim}{P}\ \delta_1^*\ \underset{\sim}{Q}$, we must have $g^{-1}(C)\ \delta_1\ g^{-1}(D)$. But g is a proximity mapping, yielding $C\ \delta_2\ D$, a contradiction.

That $\bar{g}(\underset{\sim}{X}) = \underset{\sim}{Y}$ follows from the fact that $f(Y) \subset \bar{g}(\underset{\sim}{X}) \subset \underset{\sim}{Y}$, $f(Y)$ is dense in $\underset{\sim}{Y}$, and $\bar{g}(\underset{\sim}{X})$ is compact. Therefore $\bar{g}$ is a mapping onto $\underset{\sim}{Y}$.

Finally, we show that $\bar{g}$ is unique. Suppose there is another extension $\bar{g}'$. Then there is a $\sigma \in \underset{\sim}{X}$ such that $\bar{g}(\sigma) \neq \bar{g}'(\sigma)$. Since $\underset{\sim}{Y}$ is Hausdorff and $\bar{g}$ is continuous, there is a neighbourhood $\underset{\sim}{E}$ of σ such that $\bar{g}(\underset{\sim}{E}) \cap \bar{g}'(\underset{\sim}{E}) = \phi$. Now $f(X)$ is dense in $\underset{\sim}{X}$, implying that there exits a point cluster $\sigma_x \in \underset{\sim}{E} \cap f(X)$. For such a σ_x , $\bar{g}(\sigma_x) \neq \bar{g}'(\sigma_x)$. Therefore, $\bar{g}$ and $\bar{g}'$ do not agree on $f(X)$, and hence on X .

As is well-known, compactifications of a topological space X may be partially ordered in the following manner:

> $X_1^* \geq X_2^*$ iff the identity mapping of X can be extended
> to a continuous function of X_1^* onto X_2^* .

We have already defined, in Section 2, a partial order on the set of all proximities on a set X . The following important result follows directly from Theorems (3.7) and (7.10).

(7.11) Theorem. Given any Tychonoff space, the map u of the partially ordered set $\{\delta_i : i \in I\}$ of compatible proximities onto the partially ordered set $\{X_i^* : i \in I\}$ of Smirnov compactifications is an order-isomorphism. That is, $\delta_i \geq \delta_j$ iff $X_i^* \geq X_j^*$ where $u(\delta_k) = X_k^*$ for $k \in I$.

The above theorem is actually valid for a completely regular space, as was proven by Smirnov.

Following is an analogue of Urysohn's lemma for normal spaces.

(7.12) Theorem. In a proximity space (X,δ), $A \not\delta B$ implies there exists a proximity mapping $g : X \rightarrow [0,1]$ such that $g(A) = 0$ and $g(B) = 1$.

Proof. If $A \not\delta B$, then $\bar{A} \cap \bar{B} = \emptyset$ in $\underset{\sim}{X}$. But $\underset{\sim}{X}$ is normal so that by Urysohn's lemma, there exists a continuous function $\bar{g} : \underset{\sim}{X} \rightarrow [0,1]$ such that $\bar{g}(A) = 0$ and $\bar{g}(B) = 1$. By Theorem (4.4), $\bar{g}$ is a proximity mapping. Letting $g = \bar{g}/f(X)$ and identifying X with $f(X)$, where $f(x) = \sigma_x$, we obtain the required mapping.

We now state, without proof, the analogue of Tietz's extension theorem.

(7.13) Theorem. Let A be any subspace of a proximity space X and let g be a proximity mapping from A to $[0,1]$. Then g can be extended to a proximity mapping $\bar{g} : X \rightarrow [0,1]$.

The compactification of a proximity space, which we have considered above, is a special case of the general theory of proximal extensions. We shall now consider briefly this general theory and show its relationship to the theory of compactifications.

(7.14) Definition. A proximal (or δ-) extension of a proximity space (X,δ) is a proximity space (Y,δ') such that $\overline{X} = Y$ and $\delta = \delta'/X$. A proximity space is maximal (or absolutely closed) iff it is its own δ-extension.

(7.15) Theorem. A separated proximity space (X,δ) is maximal if and only if every cluster in X is a point cluster.

Proof. If X is not maximal, there is a non-isolated point ξ which can be adjoined to it so as to obtain a proximal extension Y. Then from Theorem (5.16), there exists a unique cluster σ in X which is a subclass of the point cluster σ_ξ in Y. Obviously σ is not a point cluster in X.

Conversely, if there are clusters in X which are not point clusters, then we can construct a proximal extension of X as in Lemma (7.4). Thus X is not maximal.

(7.16) Corollary. A separated proximity space is maximal if and only if it is compact.

Every Tychonoff space has a maximal compactification, namely the Stone-Čech compactification. The following theorem characterizes those separated proximity spaces which have minimal compactifications.

(7.17) Theorem. A Tychonoff space X has a minimal compactification if and only if it is locally compact.

Proof. If X is locally compact, we may resort to the Alexandroff one-point compactification, which is the minimal compactification of X.

Conversely, suppose that X has a minimal compactification X^*. In order to show that X^* is the Alexandroff one-point compactification, suppose there are two different points ξ and η in $(X^* - X)$. We can then construct a smaller compactification, X_1^*, than X^* by "pasting" ξ and η together. Open sets in X_1^* are then those which do not contain ξ or η, and those which contain both ξ and η.

(7.18) Corollary. Every Tychonoff space X has a maximal compatible proximity. It has a minimal compatible proximity if and only if it is locally compact.

A separated proximity relation δ may be defined on a locally compact Hausdorff space by

(7.19) $A \not\delta B$ iff $\overline{A} \cap \overline{B} = \emptyset$ and either $\overline{A}$ or $\overline{B}$ is compact.

Indeed, applying Lemma (3.6), this can be shown to be the smallest compatible proximity relation possible on a locally compact Hausdorff space. We have previously seen that the largest proximity which can be defined on a Tychonoff space is $A \not\delta B$ iff A and B are functionally distinguishable. If a Tychonoff space possesses a unique proximity, clearly the maximal and minimal compactifications coincide. These considerations together with the preceding results yield the following important theorem.

(7.20) **Theorem.** A Tychonoff space X has a unique compatible proximity if and only if every pair of non-compact closed subsets A and B of X are functionally indistinguishable.

We now consider those topological spaces which have a unique compatible proximity given by

$$A \,\delta\, B \text{ iff } \overline{A} \cap \overline{B} \neq \emptyset .$$

It has already been shown (Theorem (3.7)) that compact Hausdorff spaces are of this form.

(7.21) **Definition.** A separated proximity space (X,δ) is equinormal if $\overline{A} \cap \overline{B} = \emptyset$ implies $A \not\delta B$.

It follows from Theorem (7.12) that every equinormal proximity space is normal. The converse is not true, however, since in (2.18) we have $\overline{A} \cap \overline{B} = \emptyset$ although $A \,\delta_1\, B$. The following is a generalization

of Theorem (4.4).

(7.22) Theorem. A normal separated proximity space (X,δ) is equinormal if and only if every real-valued continuous function on X is a proximity mapping.

Proof. The proof of necessity is exactly the same as that for Theorem (4.4).

To prove the converse, let X be a normal proximity space such that every real-valued continuous function on X is a proximity mapping. If $\bar{A} \cap \bar{B} = \phi$, then by Urysohn's lemma there exists a continuous function $f : X \to [0,1]$ such that $f(\bar{A}) = 0$ and $f(\bar{B}) = 1$. By hypothesis, f is also a proximity mapping and so $\bar{A} \not\delta \bar{B}$, i.e. $A \not\delta B$.

(7.23) Corollary. A normal separated proximity space (X,δ) is equinormal iff the proximity δ is induced by the Stone-Čech compactification of X.

(7.24) Corollary. Every proximity for a normal T_1-space X is equinormal iff X has a unique compactification.

§8. Proximity Weight and Compactification

In the previous section we studied the compactifications of proximity spaces and the various implications of this theory. In this section we shall define topological weight and proximity weight, and prove stronger results concerning compactifications. These developments will lead to some interesting metrization theorems concerning proximity spaces.

(8.1) **Definition.** The topological weight $w(\tau)$ of a topological space (X,τ) is the smallest cardinal number $\alpha \geq \aleph_0$ such that τ has a topological base $\underset{\sim}{B}$ with $|\underset{\sim}{B}| \leq \alpha$.

It is easy to see that $w(\tau)$ is a topological invariant.

(8.2) **Definition.** A proximity base $\underset{\sim}{P}$ for a proximity space (X,δ) is a subset of 2^X such that $A \not\delta B$ implies there are members U and V of $\underset{\sim}{P}$ such that $A \subset U$, $B \subset V$ and $U \not\delta V$.

It is obvious from Lemma (2.8) that the collection of closed sets, in a proximity space, forms a proximity base.

(8.3) **Lemma.** If $\underset{\sim}{P}$ is a proximity base, then so is $\underset{\sim}{P}_o = \{\text{Int } U : U \in \underset{\sim}{P}\}$.

Proof. If $A \not\delta B$, then by (3.4) there are sets C and D such that $A \ll C$, $B \ll D$ and $C \not\delta D$. This in turn implies the existence of members U and V of $\underset{\sim}{P}$ such that $C \subset U$, $D \subset V$ and $U \not\delta V$. But $A \subset \text{Int } C \subset \text{Int } U \subset U$ and

$B \subset \operatorname{Int} D \subset \operatorname{Int} V \subset V$. Hence $\operatorname{Int} U \not\delta \operatorname{Int} V$, and the conclusion follows.

(8.4) **Corollary.** Given a proximity space (X,δ) , $\tau(\delta)$ is a proximity base.

(8.5) **Lemma.** If $\underset{\sim}{P}$ is a proximity base of open sets, then $\underset{\sim}{P}$ is a topological base.

Proof. Let G be an open set and $x \in G$. Then $x \not\delta (X - G)$ and hence there are sets U and V in $\underset{\sim}{P}$ such that $x \in U$, $(X - G) \subset V$ and $U \not\delta V$. Also $U \cap V = \phi$, which shows that $x \in U \subset (X - V) \subset G$. Thus $\underset{\sim}{P}$ is a topological base.

(8.6) **Theorem.** Let $\underset{\sim}{B}$ be a topological base for a compact proximity space (X,δ) . Then $\underset{\sim}{B}^* = \{B : B = \bigcup_{i=1}^{n} B_i, \ B_i \in \underset{\sim}{B}\}$ is a proximity base.

Proof. From Lemma (2.8), $A \not\delta B$ implies $\overline{A} \cap \overline{B} = \phi$. Since X is regular, for each $x \in \overline{A}$ there is a $U_x \in \underset{\sim}{B}$ such that $x \in U_x \subset \overline{U}_x \subset (X - \overline{B})$. Now $\overline{A}$ is compact, so that $\overline{A} \subset \bigcup_{i=1}^{r} U_{x_i} \subset \bigcup_{i=1}^{r} \overline{U}_{x_i} \subset (X - \overline{B})$. Using a similar argument we obtain $\overline{B} \subset \bigcup_{j=1}^{s} V_{y_j} \subset \bigcup_{j=1}^{s} \overline{V}_{y_j} \subset (X - \bigcup_{i=1}^{r} \overline{U}_{x_i})$. Let $U^* = \bigcup_{i=1}^{r} U_{x_i}$ and $V^* = \bigcup_{j=1}^{s} V_{y_j}$. Then $U^* \in \underset{\sim}{B}^*$, $V^* \in \underset{\sim}{B}^*$, $A \subset U^*$, $B \subset V^*$ and $\overline{U^*} \cap \overline{V^*} = \phi$. Therefore $U^* \not\delta V^*$, and we conclude that $\underset{\sim}{B}^*$ is a proximity base.

The following result is obvious.

(8.7) Lemma. Let $\underset{\sim}{P}$ be a proximity base of X and let $Y \subset X$. Then $\underset{\sim Y}{P} = \{U \cap Y : U \in \underset{\sim}{P}\}$ is a proximity base for the subspace Y.

(8.8) Lemma. Let Y be a dense subspace of a proximity space X and let $\underset{\sim}{Q}$ be a proximity base for Y. Then $\underset{\sim}{P} = \{\overline{U} : U \in \underset{\sim}{Q}\}$ is a proximity base for X.

Proof. If A and B are subsets of X such that $A \not\delta B$, then by (3.5) there are open sets G and H such that $A \subset G$, $B \subset H$ and $G \not\delta H$. Hence $(G \cap Y) \not\delta_Y (H \cap Y)$, which implies there are members U and V of $\underset{\sim}{Q}$ such that $(G \cap Y) \subset U \subset Y$, $(H \cap Y) \subset V \subset Y$ and $U \not\delta_Y V$. Since $\overline{Y} = X$, $G \subset \overline{(G \cap Y)}$ and $H \subset \overline{(H \cap Y)}$. Thus $G \subset \overline{U}$, $H \subset \overline{V}$ and $\overline{U} \not\delta \overline{V}$.

(8.9) Definition. The proximity weight $w(\delta)$ of a proximity space (X,δ) is the smallest cardinal number $\alpha \geq \aleph_0$ such that X has a proximity base $\underset{\sim}{P}$ with $|\underset{\sim}{P}| \leq \alpha$.

It can be shown that $w(\delta)$ is a proximity invariant. Applying Corollary (8.4) and Lemma (8.5), we obtain the following result:

(8.10) Theorem. In a proximity space (X,δ) with $\tau = \tau(\delta)$,

$$w(\tau) \leq w(\delta) \leq 2^{w(\tau)} .$$

Using this theorem together with Theorem (8.6), we obtain:

(8.11) Corollary. For a compact proximity space (X,δ) , $w(\tau) = w(\delta)$.

(8.12) Remark. It should be pointed out that the converse of (8.11) is false. For an infinite discrete space, $w(\tau) = w(\delta)$ although the space is not compact.

The next lemma follows easily from Lemma (8.7) and (8.8).

(8.13) Lemma. If (Y,δ) is a subspace of (X,δ^*) , then $w(\delta) \leq w(\delta^*)$. If Y is dense in X , their proximity weights are equal.

We shall now consider a strengthened form of Theorem (7.7): namely, that the compactification of separated proximity space can be effected so as to preserve the proximity weight.

(8.14) Theorem. Every separated proximity space (X,δ) of proximity weight θ can be embedded δ-homeomorphically as a dense subspace of a compact Hausdorff space X^* of the same proximity weight θ . X^* is unique up to δ-homeomorphism.

This theorem will be proved below in a sequence of lemmas. The construction used here reminds one of a similar one used in constructing the Stone-Čech compactification of a Tychonoff space.

Let $\underset{\sim}{P}$ be a proximity base for X such that $|\underset{\sim}{P}| \leq \theta$, and let $\underset{\sim}{C} = \{(U,V) : U,V \in \underset{\sim}{P} \text{ and } U \not\delta V\}$. Then $|\underset{\sim}{C}| \leq \theta$. We know from Theorem (7.12) that for each $\gamma = (U,V) \in \underset{\sim}{C}$, there exists a proximity mapping $f_\gamma : X \to [0,1]$ such that $f_\gamma(U) = 0$ and $f_\gamma(V) = 1$. Now let $\underset{\sim}{F} = \{f_\gamma : \gamma = (U,V) \in \underset{\sim}{C}\}$ and let $K = \prod_{\gamma \in \underset{\sim}{C}} [0,1]$ be the Tychonoff

cube of $|\underset{\sim}{C}|$ -copies of $[0,1]$. Then K is a compact Hausdorff space and so, by Theorem (3.7), has a unique proximity δ^* given by $A \delta^* B$ iff $\bar{A} \cap \bar{B} \neq \phi$. Define $f : X \to K$ by $f(x) = (f_\gamma(x))$.

(8.15) Lemma. The function f is one-to-one.

Proof. If $x \neq y$, then $x \not\delta y$. Therefore there are members U and V of $\underset{\sim}{C}$ such that $x \in U$, $y \in V$, $f_\gamma(U) = 0$ and $f_\gamma(V) = 1$, where $\gamma = (U,V)$. Hence $f(x) \neq f(y)$.

(8.16) Lemma. $f^{-1} : f(X) \to X$ is a proximity mapping.

Proof. If A and B are subsets of X such that $A \not\delta B$, then there are members U and V in $\underset{\sim}{P}$ such that $A \subset U$, $B \subset V$ and $U \not\delta V$. Let $\gamma_0 = (U,V)$. Now $f(A) \subset \mathbb{P}_{\gamma \in \underset{\sim}{C}} U^\gamma$ and $f(B) \subset \mathbb{P}_{\gamma \in \underset{\sim}{C}} V^\gamma$, where $U^{\gamma_0} = \{0\}$, $V^{\gamma_0} = \{1\}$ and $U^\gamma = V^\gamma = [0,1]$ for all other γ . The sets containing $f(A)$ and $f(B)$ are closed and disjoint, showing that $f(A) \not\delta^* f(B)$. Thus f^{-1} is a proximity mapping.

(8.17) Lemma. The function $f : X \to f(X)$ is a proximity mapping.

Proof. Since $f_\gamma : X \to [0,1]$ is a proximity mapping, Theorem (7.10) assures the existence of a unique extension $\bar{f}_\gamma : \underset{\sim}{X} \to [0,1]$ where $\bar{f}_\gamma$ is a proximity mapping on the Smirnov compactification $\underset{\sim}{X}$ of X . Since $\bar{f}_\gamma$ is continuous

for each $\gamma \in \underset{\sim}{C}$, $\overline{f} : \underset{\sim}{X} \to [0,1]$, where $\overline{f}(x) = (\overline{f}_\gamma(x))$, is continuous. Since $\underset{\sim}{X}$ is compact, $\overline{f}$ is a proximity mapping. But this implies that f is a proximity mapping, for $f = \overline{f}/X$.

(8.18) Definition. A proximity space is said to be metrizable iff it is δ-homeomorphic to a metric proximity space.

(8.19) Theorem. A proximity space (X,δ) is proximally homeomorphic to a totally bounded metric space if and only if its proximity weight $w(\delta) = \aleph_0$.

Proof. Suppose $w(\delta) = \aleph_0$. By Theorem (8.14), the compactification $\underset{\sim}{X}$ of X has proximity weight $\aleph_0$, which equals the topological weight $w(\tau)$. But a Tychonoff topological space with $w(\tau) = \aleph_0$ is metrizable. Now a compact metric space is totally bounded and total boundedness is hereditary, showing that X is a totally bounded metric space.

Conversely, if (X,δ) is proximally homeomorphic to a totally bounded metric space (Y,d) then Y^* , the completion of Y , is compact and hence separable. A separable metric space has a countable base, and so its topological weight, and hence the proximity weight of Y , is $\aleph_0$. Thus the proximity weight of X if $\aleph_0$.

(8.20) Corollary. Every proximity space with proximity weight $\aleph_0$ is metrizable.

(8.21) **Theorem.** The proximity weight of a metric proximity space is $\aleph_0$ if and only if it is totally bounded.

Proof. Sufficiency follows from Theorem (8.19). On the other hand, if a metric space (Y,d) has proximity weight $\aleph_0$, then Y is δ-homeomorphic to a totally bounded metric space (Y',d'). Since Y and Y' are uniformly isomorphic, Y too is totally bounded.

The above theorem is a particular case of a more general result concerning the proximity weight of a metric space.

(8.22) **Definition.** A subset E of a metric space (Y,d) is ε-discrete iff $d(x,y) \geq \varepsilon$ for every pair of distinct points x,y of E.

Given $\varepsilon > 0$, there exists, by Tukey's lemma, a maximal ε-discrete subset E_ε of Y; that is, $E_\varepsilon \subset E \subset Y$ and E being ε-discrete implies $E = E_\varepsilon$.

(8.23) **Theorem.** Let (Y,d) be a metric space of topological weight ψ and proximity weight θ. For $n \in N$, let E_n be a maximal $\frac{1}{n}$-discrete subset of Y and let $|E_n| = \sigma_n$. Then

$$\psi = \sum_{n=1}^{\infty} \sigma_n$$

$$\theta = \sum_{n=1}^{\infty} 2^{\sigma_n} .$$

Proof. The family $\{S(x, \frac{1}{2n}) : x \in E_n\}$ consists of pairwise disjoint non-empty open sets in Y. The cardinality of this family is σ_n, so that $\sigma_n \leq \psi$. Hence $\sum_{n=1}^{\infty} \sigma_n \leq \aleph_0 \psi \leq \aleph_0$. On the other hand, $\{S(x, \frac{1}{n}) : x \in E_n, n \in N\}$ covers Y and forms a topological base. The cardinality of this family is $\leq \sum_{n=1}^{\infty} \sigma_n$, so that $\psi \leq \sum_{n=1}^{\infty} \sigma_n$. Thus $\psi = \sum_{n=1}^{\infty} \sigma_n$.

Every pair of disjoint subsets of E_n are not near, so that every proximity base of E_n must contain all subsets of E_n. Thus the proximity weight of E_n is $\geq 2^{\sigma_n}$, showing that $\theta \geq \aleph_0\, 2^{\sigma_n} \geq \sum_{n=1}^{\infty} 2^{\sigma_n}$. On the other hand, if the subsets A and B of Y are not near, then $D(A,B) = \varepsilon > 0$. Choose n such that $\frac{1}{n} < \varepsilon/5$. Let $U = \bigcup_{x \in A} S(x, \frac{1}{n})$ and $V = \bigcup_{x \in B} S(x, \frac{1}{n})$. Then $A \subset U$ and $B \subset V$ and $D(U,V) \geq 3\varepsilon/5 > 0$, implying $U \not\delta^{*} V$. So a proximity base for Y can be constructed by forming unions of spheres $S(x, \frac{1}{n})$ with centres in E_n and all possible subsets of E_n for $n \in N$. This proximity base has cardinality $\leq \sum_{n=1}^{\infty} 2^{\sigma_n}$, i.e. $\theta \leq \sum_{n=1}^{\infty} 2^{\sigma_n}$. Thus $\theta = \sum_{n=1}^{\infty} 2^{\sigma_n}$.

(8.24) Corollary. A metric proximity space has proximity weight $\aleph_0$ or $\geq 2^{\aleph_0}$ according as it is totally bounded or not.

Using techniques similar to the above, the following results can be proved. However, the proofs are omitted here.

(8.25) Theorem. If $\sigma_n = \psi$ for at least one $n \in N$, then $\theta = 2^{\psi}$. This is the particular case when $\psi = \eta^{\aleph_0}$, where η is an arbitrary cardinal number.

(8.26) Theorem. For a metric proximity space, $\theta = 2^{\psi}$ or ψ according as there exists or does not exist an $n \in N$ such that $\sigma_n = \psi$.

§9. Local Proximity Spaces.

In Section 7 it was shown that a compact Hausdorff space $\underset{\sim}{X}$ is (uniquely) determined by a dense proximity space (X,δ), and that $A \,\delta\, B$ in X iff $\overline{A} \cap \overline{B} \neq \emptyset$ where the closures are taken in $\underset{\sim}{X}$. Now a locally compact Hausdorff space Y is determined by a dense subspace X when we know not only the proximity of X, but also which sets in X have compact closures in Y. (Such sets may be called bounded.) Boundedness in topological spaces has been studied axiomatically by various authors since 1939. This section is devoted to a treatment of the interrelationships existing between proximities and bounded sets, and eventually leads to the concept of a local proximity space. It will be seen that such spaces can be embedded as dense subspaces of locally compact Hausdorff spaces.

Firstly, however, we shall pursue a brief study of boundedness in topological spaces.

(9.1) **Definition.** A non-empty collection $\underset{\sim}{B}$ of subsets of a topological space X is called a boundedness in X iff:

(i) $A \in \underset{\sim}{B}$ and $B \subset A$ implies $B \in \underset{\sim}{B}$.

(ii) $\{A_i : i = 1,\ldots,n\} \subset \underset{\sim}{B}$ implies $\bigcup_{i=1}^{n} A_i \in \underset{\sim}{B}$.

Elements of $\underset{\sim}{B}$ are called bounded sets.

(9.2) **Remarks.** (i) If (X,d) is a metric space, we may define a metric boundedness

$$\underset{\sim}{B}_d = \{A \subset X : \sup_{x,y \in A} d(x,y) < \infty\} \quad .$$

It is clear that a metric boundedness is a boundedness.

(ii) Given any topological space X, we may define a boundedness:

$$\underset{\sim}{B} = \{A \subset X : A \text{ is finite}\} \quad .$$

(iii) If $\underset{\sim}{B}$ is any boundedness, $\phi \in \underset{\sim}{B}$.

(iv) The intersection of a non-empty collection of bounded sets is bounded.

(v) If X is bounded, then every subset of X is bounded.

(9.3) Lemma. If $\underset{\sim}{B}$ is a boundedness in X and X is unbounded, then

$$\underset{\sim}{F} = \{F : (X - F) \in \underset{\sim}{B}\}$$

is a filter.

Proof. Since X is unbounded, each $F \in \underset{\sim}{F}$ is non-empty. If F_1 and F_2 both belong to $\underset{\sim}{F}$, then $(X - F_1)$ and $(X - F_2)$ both belong to $\underset{\sim}{B}$. Then by 9.1 (ii), $(X - F_1) \cup (X - F_2) = [X - (F_1 \cap F_2)] \in \underset{\sim}{B}$; in other words, $F_1 \cap F_2 \in \underset{\sim}{F}$. If $F \in \underset{\sim}{F}$ and $F \subset G$, then $(X - G) \subset (X - F)$. Since $(X - F) \in \underset{\sim}{B}$, 9.1 (i) yields $G \in \underset{\sim}{F}$. Thus $\underset{\sim}{F}$ is a filter.

(9.4) *Corollary*. If for every subset E of X either $E \in \underset{\sim}{B}$ or $(X - E) \in \underset{\sim}{B}$, then $\underset{\sim}{F}$ is an ultrafilter.

Throughout the sequel, we shall always suppose that $\underset{\sim}{B}$ is a boundedness in a topological space X.

(9.5) *Definition*. A topological space is *locally bounded* iff each point has a bounded neighbourhood.

The following result is proved by using usual techniques.

(9.6) *Theorem*. A compact subset of a locally bounded topological space is bounded.

(9.7) *Definition*. A topological space with a boundedness $\underset{\sim}{B}$ is *compactly bounded* iff

$$\underset{\sim}{B} = \{B \subset X : \overline{B} \text{ is compact}\} .$$

The following theorem is obvious.

(9.8) *Theorem*. A Hausdorff space X is locally compact if and only if X being compactly bounded implies X is locally bounded.

(9.9) *Definition*. X is *boundedly compact* ("Montel" space) iff every closed bounded subset of X is compact.

In proving the following theorem and corollary, we use two well-known characterizations of compactness: A subset E of a topological

space X is compact iff every filter base in E has a cluster point in E iff every ultrafilter in E converges to a point in E .

(9.10) Theorem. X is boundedly compact if and only if every closed bounded filter base (i.e. a filter base consisting of closed bounded sets) has a cluster point.

Proof. Suppose that X is boundedly compact and let $\underset{\sim}{F}$ be any closed bounded filter base in X . Given any $F_0 \in \underset{\sim}{F}$, consider the filter base $\underset{\sim}{F}_0 = \{F \cap F_0 : F \in \underset{\sim}{F}\}$. Since F_0 is compact, $\underset{\sim}{F}_0$ has a cluster point $x_0 \in F_0$. Clearly x_0 is also a cluster point of $\underset{\sim}{F}$, and necessity is proved.

Now let B be any closed bounded subset of X , and consider any filter base $\underset{\sim}{F}$ in B . Then $\overline{\underset{\sim}{F}} = \{\overline{F} : F \in \underset{\sim}{F}\}$ is a closed bounded filter base in $B \subset X$ and, by hypothesis, must have a cluster point x_0 . In fact $x_0 \in \overline{F}$ for each $F \in \underset{\sim}{F}$, so that x_0 is also a cluster point of $\underset{\sim}{F}$. So every filter base in B has a cluster point, implying that B is compact. Thus X is boundedly compact.

(9.11) Corollary. X is boundedly compact iff every ultrafilter containing a closed bounded filter base converges.

(9.12) Definition. A local proximity space is a triple $(X,\alpha,\underset{\sim}{B})$ where X is a set, $\underset{\sim}{B}$ is a boundedness in X and α is a binary relation on 2^X satisfying (1.1), (1.2), (1.3), (1.5), in addition to the two

following axioms:

(a) Let $A \subset X$ and $B \in \underset{\sim}{B}$. If for every $C \in \underset{\sim}{B}$ either $A \alpha C$ or $(X - C) \alpha B$, then $A \alpha B$.

(b) If $A \alpha B$, then there is a set $D \in \underset{\sim}{B}$ such that $D \subset B$ and $A \alpha D$.

(9.13) Remarks. (i) If X is bounded, then α is a proximity on X .

(ii) If (X,δ) is a proximity space and $\underset{\sim}{B} = 2^X$, then $(X,\delta,\underset{\sim}{B})$ is a local proximity space.

(iii) From the definition of a boundedness and 9.12 (b), we deduce that every finite subset of X is bounded.

(iv) Let X denote the positive real numbers with the usual metric proximity δ . Let $\underset{\sim}{F}$ be the filter generated by the filter base consisting of all right rays. Let $\underset{\sim}{B}$ consist of all complements of members of $\underset{\sim}{F}$, and define:

$$A \alpha B \quad \text{iff} \quad (E \cap A) \, \delta \, (E \cap B) \quad \text{for some} \quad E \in \underset{\sim}{B} .$$

It will be shown in Theorem (9.16) that, since $\underset{\sim}{F}$ is a free round filter, $(X,\alpha,\underset{\sim}{B})$ is a local proximity space.

(v) $B \in \underset{\sim}{B}$ implies the existence of a $C \in \underset{\sim}{B}$ such that $B \not\alpha (X - C)$.

(vi) As a consequence of 9.12 (a) we obtain the following, where $<<$ is defined with respect to α :

If $A \in \underset{\sim}{B}$ and $A << C$, there exists a $B \in \underset{\sim}{B}$ such

that $A \ll B \ll C$.

(9.14) Definition. A filter $\underset{\sim}{F}$ is free iff $\bigcap_{F \in \underset{\sim}{F}} F = \emptyset$.

(9.15) Definition. Given a local proximity space $(X,\alpha,\underset{\sim}{B})$, a binary relation β is said to agree locally with α iff β and α agree whenever either of the sets involved is bounded.

(9.16) Theorem. Let $\underset{\sim}{F}$ be a free round filter in a proximity space (X,δ) . Define:

(a) $\underset{\sim}{B} = \{B \subset X : (X - B) \in \underset{\sim}{F}\}$.

(b) $A \,\alpha\, B$ iff $(E \cap A)\, \delta\, (E \cap B)$ for some $E \in \underset{\sim}{B}$.

Then $(X,\alpha,\underset{\sim}{B})$ is a local proximity space and α agrees with δ locally.

Proof. That $\underset{\sim}{B}$ is a boundedness is immediate from (a). To prove that α agrees with δ locally, suppose $A\, \delta\, B$ where $B \in \underset{\sim}{B}$. Since $\underset{\sim}{F}$ is a round filter, the duality between $\underset{\sim}{F}$ and $\underset{\sim}{B}$ yields an $E \in \underset{\sim}{B}$ such that $B \ll E$ relative to δ . Now $B \not\delta [(X - E) \cap A]$, so that $B\, \delta\, (E \cap A)$. Since $(E \cap B) = B$, $(E \cap A)\, \delta\, (E \cap B)$. Then by definition, $A \,\alpha\, B$. That $A \,\alpha\, B$ implies $A\, \delta\, B$ is obvious.

The definition of α and the fact that $\{x\} \in \underset{\sim}{B}$ for every $x \in X$ together imply that α satisfies (1.5). The remainder of the axioms for a local proximity space, with the possible exception of 9.12 (a), follow easily. We now

prove the contrapositive of 9.12 (a), namely $B \in \underset{\sim}{B}$ and $B \ll D$ implies there is a $C \in \underset{\sim}{B}$ such that $B \ll C \ll D$ relative to α. Let $B \in \underset{\sim}{B}$ and $B \ll D$ relative to α. Then $B \ll D$ relative to δ also, for α and δ agree locally. Since δ is a proximity, there exists an A such that $B \ll A \ll D$ relative to δ. As before, choose an $E \in \underset{\sim}{B}$ such that $B \ll E$. Letting $C = E \cap A$, we obtain the required required result since $C \in \underset{\sim}{B}$ and δ agrees locally with α.

The next result shows how a proximity may be defined on a local proximity space. This is valuable since it enables one to derive properties of local proximity spaces from known properties of proximity spaces. The proof is trivial.

(9.17) Theorem. Let $(X,\alpha,\underset{\sim}{B})$ be a local proximity space, and define δ by:

$$A \;\delta\; B \quad \text{iff} \quad A \;\alpha\; B, \text{ or } A \notin \underset{\sim}{B} \text{ and } B \notin \underset{\sim}{B} .$$

Then δ, called the Alexandroff extension of α, is a proximity on X.

The next corollary follows from the fact that, due to 9.13 (iii), $x \;\delta\; A$ iff $x \;\alpha\; A$.

(9.18) Corollary. (i) If for $A \subset X$, we define $\overline{A} = \{x : x \;\alpha\; A\}$, then X becomes a completely regular topological space.

(ii) $A \;\alpha\; B$ iff $\overline{A} \;\alpha\; \overline{B}$.

Observe that the proximity δ defined in the above theorem is the largest binary relation on 2^X which agrees locally with α .

(9.19) *Lemma*. Every local proximity space $(X,\alpha,\underset{\sim}{B})$ is locally bounded.

Proof. This lemma follows directly from 9.13 (iii) and the contrapositive of 9.12 (a), namely $B \in \underset{\sim}{B}$ and $B \ll D$ implies there is a $C \in \underset{\sim}{B}$ such that $B \ll C \ll D$ relative to α .

(9.20) *Lemma*. In a local proximity space $(X,\alpha,\underset{\sim}{B})$, the closure of a bounded set is bounded.

Proof. Let $B \in \underset{\sim}{B}$ and consider $\overline{B}$. By 9.13 (v), there exists a $C \in \underset{\sim}{B}$ such that $B \not\alpha (X - C)$. Hence by 9.18 (ii), $\overline{B} \not\alpha (X - C)$. Therefore $\overline{B} \subset C$, and $\overline{B} \in \underset{\sim}{B}$.

(9.21) *Lemma*. Every compact subset of a local proximity space is bounded.

Proof. Since every local proximity space is locally bounded, we simply apply Theorem (9.6).

(9.22) *Lemma*. Let $(X,\alpha,\underset{\sim}{B})$ be a local proximity space and define δ as in Theorem (9.17). If X is unbounded, then $\sigma = \{A : A \notin \underset{\sim}{B}\}$ is a cluster in (X,δ) .

Proof. By the definition of δ , any two sets in σ are near. From 9.13 (v), it follows that if $A \,\delta\, C$ for every

$C \in \sigma$, then $A \in \sigma$. If $A \notin \sigma$ then $B \notin \sigma$, then $A \in \underset{\sim}{B}$ and $B \in \underset{\sim}{B}$. Hence $(A \cup B) \in \underset{\sim}{B}$, i.e. $(A \cup B) \notin \sigma$. So σ is a cluster.

We now turn our attention to the local compactification of a local proximity space. This could be done by considering the space of all bounded clusters (i.e. clusters determined by bounded ultrafilters) and defining a suitable topology on this space. Since a compactification of a proximity space has already been constructed in Section 7, and by Theorem (9.17) a proximity space is always associated with a local proximity space, this local compactification can be performed more easily.

(9.23) Theorem. Given a local proximity space $(X, \alpha, \underset{\sim}{B})$, there exists a locally compact Hausdorff space L and a one-to-one map $f : X \to L$ satisfying the following conditions:

(i) $A \alpha B$ iff $\overline{f(A)} \cap \overline{f(B)} \neq \emptyset$ in L.

(ii) $B \in \underset{\sim}{B}$ iff $\overline{f(B)}$ is compact in L.

(iii) $\overline{f(X)} = L$.

Such a local compactification is unique; that is, given $f_i : X \to L_i$ $(i = 1,2)$ satisfying the above conditions, there exists a homeomorphism h from L_1 onto L_2 such that $f_2 = h \circ f_1$. The function f is onto iff X is boundedly compact. L is compact iff X is bounded. Conversely, if we are given a mapping $f : X \to L$ and α, $\underset{\sim}{B}$ are defined by (i) and (ii) above, then $(X, \alpha, \underset{\sim}{B})$ is a local proximity space.

Proof. Let δ be the Alexandroff extension of α (Theorem (9.17)). Let $(f,\underset{\sim}{X})$ be the Smirnov compactification of (X,δ) (Theorem (7.7)); i.e. $\underset{\sim}{X}$ is a compact Hausdorff space, $A \,\delta\, B$ iff $\overline{f(A)} \cap \overline{f(B)} \neq \phi$, and $\overline{f(X)} = \underset{\sim}{X}$. If X is bounded, $\delta = \alpha$ and $L = \underset{\sim}{X}$. So we need only consider the case for which $X \notin \underset{\sim}{B}$. Now $\underset{\sim}{X}$ consists of all clusters in X and so, by Lemma (9.22), $\sigma = \{A : A \notin \underset{\sim}{B}\} \in \underset{\sim}{X}$. Clearly $\sigma = \{A : \sigma \in \overline{f(A)}\}$. In other words, $B \in \underset{\sim}{B}$ iff $\overline{f(B)} \subset (\underset{\sim}{X} - \{\sigma\})$. Let $L = \underset{\sim}{X} - \{x\}$. Then L is locally compact and Hausdorff. Since $\{x\} \in \underset{\sim}{B}$ for each $x \in X$, $f(X) \subset L$. One can now readily see that 9.23 (ii) and (iii) follow. Now α and δ are locally equivalent and, by 9.12 (b), if $A \,\alpha\, B$ we may take $A \in \underset{\sim}{B}$ and $B \in \underset{\sim}{B}$. As a result we obtain 9.23 (i).

(9.24) For every $q \in L$, there exists a $B \in \underset{\sim}{B}$ such that $q \in \overline{f(B)}$.

Since L is locally compact, q has a compact neighbourhood Q. By 9.23 (ii), $B = f^{-1}(Q)$ is bounded and $q \in \overline{f(B)}$.

We now prove uniqueness. Let $\underset{\sim}{X}_i = L_i \cup \{\sigma_i\}$ be the one-point compactification of L_i. Then $(f_i,\underset{\sim}{X}_i)$ is the Smirnov compactification of (X,δ). By Theorem (7.7), the Smirnov compactification is unique and there is a homeomorphism h from $\underset{\sim}{X}_1$ onto $\underset{\sim}{X}_2$ such that $h \circ f_1 = f_2$. It suffices to show that $h(\sigma_1) = \sigma_2$. If B is a bounded

set in X, $\overline{f_1(B)}$ is a compact subset of L_1 which does not contain σ_1. Since h is a homeomorphism, $h(\sigma_1) \notin h(\overline{f_1(B)}) = \overline{f_2(B)}$. Hence by (9.24), $h(\sigma_1) \notin L_2$, i.e. $h(\sigma_1) = \sigma_2$. That f is one-to-one follows from 9.23 (i).

If f is onto and B is a closed bounded set in X, we must show that B is compact. By 9.23 (ii), it suffices to show that $f(B)$ is closed in $f(X)$. Now by 9.23 (i), $f(x) \in \overline{f(B)}$ iff $x \in \overline{B}$. Since B is closed, $f(x) \in \overline{f(B)}$ iff $x \in B$, i.e. $f(x) \in f(B)$. Hence $f(B)$ is closed.

Conversely, suppose that X is boundedly compact and that $q \in L$. Choose B as in (9.24). We may assume that B is closed or, in other words, compact. But f is continuous by 9.23 (i), so that $f(B)$ is also compact. Hence $q \in f(B)$, and f is onto.

The proof of the remainder of the theorem is routine.

We next prove an analogue of Theorem (7.10) on extensions of proximity mappings.

(9.25) Definition. Let $(X_i, \alpha_i, \underset{\sim}{B}_i)$ $(i = 1,2)$ be two local proximity spaces and $f : X_1 \rightarrow X_2$ be a function. Then f is a local proximity mapping iff:

(i) $A \, \alpha_1 \, B$ implies $f(A) \, \alpha_2 \, f(B)$.

(ii) $B \in \underset{\sim}{B}_1$ implies $f(B) \in \underset{\sim}{B}_2$.

It is easy to see that the composition of two local proximity mappings is a local proximity mapping. Moreover, a local proximity mapping carries bounded clusters into bounded clusters.

(9.26) Theorem. Let $(X_i, \alpha_i, \underset{\sim}{B}_i)$ $(i = 1,2)$ be two local proximity spaces with local compactifications (f_i, L_i) $(i = 1,2)$. If $f : X_1 \rightarrow X_2$ is a local proximity mapping, then there exists a unique continuous mapping $h : L_1 \rightarrow L_2$ such that $h \circ f_1 = f_2 \circ f$. Conversely, if $h : X_1 \rightarrow X_2$ is continuous and $h \circ f_1(X_1) \subset f_2(X_2)$, then the mapping $f = f_2^{-1} \circ h \circ f_1 :$ $X_1 \rightarrow X_2$ is a local proximity mapping.

Proof. The details of this proof are left to the reader. One first extends the function f to a function $\overline{f} : \underset{\sim}{X}_1 \rightarrow \underset{\sim}{X}_2$, and then takes the restriction of $\overline{f}$ to L_1 as h.

NOTES

§5. A thorough treatment of ultrafilters may be found in Gaal [C], for example. Clusters were introduced by Leader [48]. Relationships between clusters and ultrafilters are given in Leader [49] and Mrówka [67], the latter presenting the axiomatic characterization of the family of all clusters in a proximity space. The treatment presented here is due to Warrack and Naimpally [105]. A weak form of Theorem (5.8) is implicitly contained in the aforementioned paper of Mrówka.

§6. This section owes its existence to the work of Leader [50]. An excellent survey of centered systems in topological spaces has recently been published by Iliadis and Fomin [G]. The centered system is a subbase for a filter, and is generally used by the Russian school in place of the filter. The concept of an end was originated by Alexandroff [A], while both Freudenthal [B] and Alexandroff [A] defined a round filter. Smirnov [88] then used these devices in his proximal extension theory. Alexandroff originally defined an end in a Tychonoff space X using such terms as "centered system" and "completely regular system". In filter terminology, he would define an end to be a maximal completely regular filter, the latter being a maximal filter with an open base $\underset{\sim}{B}$ such that for each $B \in \underset{\sim}{B}$, there exists an $A \in \underset{\sim}{B}$ with $A \subset B$ and a continuous function $f : X \to [0,1]$ such that $f(\overline{A}) = 0$ and $f(X - B) = 1$. Clearly, this definition of an end coincides with that of Smirnov (see (6.4)) when the proximity in question is that defined by (1.11).

§7. A general treatment of compactifications can be found in Kelley [H]. For an account of n-point compactifications, the reader is referred to Magill [I]. The results on proximal extensions are due to Smirnov [88], who was the first to expose the relationship between proximities and compactifications, which he effected using ends. The construction of the Smirnov compactification presented here is a modification of the method of Leader [48]. Equinormal proximity spaces were first investigated by Pervin [78]. For a discussion of uniformizable spaces with a unique uniformity (and hence a unique proximity), refer to Gál [D].

§8. The material in this section was taken from Császár and Mrówka [13]. Taking advantage of the Smirnov compactification constructed in the previous section, a few proofs have been simplified. The topological theorem used in the proof of (8.17) can be found on Page 91 of Kelley [H].

§9. A comprehensive study of boundedness, on which is based the first part of this section, was carried out by Hu [F]. Local proximity spaces were recently defined by Leader [54].

CHAPTER III

PROXIMITY AND UNIFORMITY

§10. Proximity Induced by a Uniformity.

In this chapter, we shall study questions concerning the relationship between uniform structures and proximity structures. Proximity structures lie between topological structures and uniform structures in the sense that all topological invariants are proximity invariants and all proximity invariants are uniform invariants; however some uniform invariants, such as total boundedness and completeness, are not proximity invariants.

Uniform structures were first defined by Weil as collections of subsets of $X \times X$ satisfying certain conditions. Tukey later provided an alternate description of a uniform structure, regarding it as a collection of covers of X satisfying certain conditions. Although we shall adhere to the Weil approach here, it should be noted the Tukey's method is widely used by the Russian school.

(10.1) Definition. A uniform structure (or uniformity) $\underset{\sim}{U}$ on a set X is a collection of subsets (called entourages) of $X \times X$ satisfying the following conditions:

(i) Every entourage contains the diagonal $\Delta = \{(x,x) : x \in X\}$.

(ii) If $U \in \underset{\sim}{U}$ and $V \in \underset{\sim}{U}$, then $U \cap V \in \underset{\sim}{U}$.

(iii) Given $U \in \underset{\sim}{U}$, there exists a $V \in \underset{\sim}{U}$ such that $V \circ V \subset U$ (triangle inequality).

(iv) If $U \in \underset{\sim}{U}$ and $U \subset V \subset X \times X$, then $V \in \underset{\sim}{U}$.

(v) If $U \in \underset{\sim}{U}$, then $U^{-1} \in \underset{\sim}{U}$.

Just as we defined a (pseudo-) metric proximity in Section 1, it is possible to define a (pseudo-) metric uniformity. Given a (pseudo-) metric d on X , let $U_\varepsilon = \{(x,y) \in X \times X : d(x,y) < \varepsilon\}$. Then the collection $\{U_\varepsilon : \varepsilon > o\}$ is a base for the (pseudo-) metric uniformity induced by d .

It can be shown that for each $x \in X$, $\{U[x] : U \in \underset{\sim}{U}\}$ is a neighbourhood filter. Thus $\underset{\sim}{U}$ generates a topology $\tau = \tau(\underset{\sim}{U})$ on X . As is well-known, this topology is always completely regular.

If $\underset{\sim}{U}$ satisfies the additional condition

(vi) $\bigcap_{U \in \underset{\sim}{U}} U = \Delta$,

then $\underset{\sim}{U}$ is called a Hausdorff or separated uniformity. In this case, $\tau(\underset{\sim}{U})$ is Tychonoff. Conversely, every (Tychonoff) completely regular space (X,τ) has a compatible (separated) uniformity, i.e. a uniformity $\underset{\sim}{U}$ such that $\tau = \tau(\underset{\sim}{U})$. Every uniformity has a base consisting of open (closed) symmetric members, and it is frequently more convenient to work with such a base for $\underset{\sim}{U}$ rather than with $\underset{\sim}{U}$ itself.

(10.2) **Theorem.** Every uniform space $(X,\underset{\sim}{U})$ has an associated proximity $\delta = \delta(\underset{\sim}{U})$ defined by

(10.3) $$A \,\delta\, B \text{ iff } (A \times B) \cap U \neq \emptyset \text{ for every } U \in \underset{\sim}{U}\,.$$

Furthermore, $\tau(\underset{\sim}{U}) = \tau(\delta)$. If $\underset{\sim}{U}$ is separated, then so is $\delta(\underset{\sim}{U})$.

Proof. All the axioms for a proximity, except perhaps (1.4), are easily verified. To verify (1.4), suppose $A \not\delta B$. Then there is an entourage U and that $(A \times B) \cap U = \emptyset$. Hence by 10.1 (v), there exists an entourage V such that $V \circ V \subset U$. Let $E = V[B]$. Then $(A \times E) \cap V = \emptyset$ and $((X - E) \times B) \cap V = \emptyset$, i.e. $A \not\delta E$ and $(X - E) \not\delta B$. To show that $\tau(\delta) = \tau(\underset{\sim}{U})$, we observe that x is in the $\tau(\underset{\sim}{U})$ - closure of A iff $x \in U[A]$ for every entourage U iff $(x \times A) \cap U \neq \emptyset$ for every entourage U iff $x \,\delta\, A$, i.e. x is in the $\tau(\delta)$-closure of A. Finally, suppose that $\underset{\sim}{U}$ is separated. If $x \,\delta\, y$, then $(x,y) \cap U \neq \emptyset$ for every entourage U. In particular, $(x,y) \cap \Delta \neq \emptyset$, so that $x = y$. Thus δ is separated.

(10.4) Remarks. (i) Instead of using (10.3), δ could equivalently be defined by

$$A \,\delta\, B \text{ iff } U[A] \cap U[B] \neq \emptyset \text{ for every } U \in \underset{\sim}{U}\,,$$

or

$$A \,\delta\, B \text{ iff } U[A] \cap B \neq \emptyset \text{ for every } U \in \underset{\sim}{U}\,.$$

Definition (10.3) has been chosen and care has been taken in writing the above proof, because the proof in this form will essentially go through if we are dealing with quasi-uniformities and quasi-proximities.

(ii) Two different uniformities on X may induce the same proximity δ on X . For example, let X be the real line and $\underset{\sim}{U}_1$ be the usual metric uniformity. Let $\underset{\sim}{U}_2$ be the subspace uniformity on X induced by the (unique) uniformity of its Smirnov compactification. Clearly, $\underset{\sim}{U}_1$ and $\underset{\sim}{U}_2$ induce the same (metric) proximity on X . However, since $\underset{\sim}{U}_1$ is not totally bounded whereas $\underset{\sim}{U}_2$ is totally bounded, $\underset{\sim}{U}_1$ and $\underset{\sim}{U}_2$ are different.

(iii) It is possible for different uniformities on X to induce the same topology and yet induce different proximities on X . This becomes immediately clear by recalling the proximities defined in Remarks (2.18) and considering the subspace uniformities induced by the Smirnov compactifications of X corresponding to δ_1 and δ_2 .

(iv) Since every (Tychonoff) completely regular space has a compatible (separated) uniformity, the above theorem gives an alternate proof of the fact that every (Tychonoff) completely regular space has a compatible (separated) proximity. The reader should note the incompatible use of the term "compatible". It is hoped that in each case the meaning will be clear from the context.

(v) It should be noted that the triangle inequality corresponds to the Strong Axiom.

Given a subset A of X and an entourage U , U[A] may be called a "uniform neighbourhood" of A . The following result explains the application of this term to a subset B when B $\gg$ A .

(10.5) Theorem. Let $(X,\underset{\sim}{U})$ be a uniform space and let $\delta = \delta(\underset{\sim}{U})$. Then $A \ll B$ if and only if there is an entourage U such that $U[A] \subset B$.

Proof. $A \ll B$ iff $A \not\delta (X - B)$ iff $(A \times (X - B)) \cap U = \emptyset$ for some $U \in \underset{\sim}{U}$. But the last statment is equivalent to $U[A] \subset B$.

(10.6) Remark. The above result suggests an alternate definition of the associated proximity relation, as $A \ll B$ iff $U[A] \subset B$ for some $U \in \underset{\sim}{U}$.

(10.7) Theorem. If $\underset{\sim}{U}_1 \subset \underset{\sim}{U}_2$, then $\delta_1 < \delta_2$ where $\delta_i = \delta(\underset{\sim}{U}_i)$ for $i = 1,2$.

Proof. $A \not\delta_1 B$ implies the existence of a $U_1 \in \underset{\sim}{U}_1$ such that $(A \times B) \cap U_1 = \emptyset$. But $U_1 \in \underset{\sim}{U}_2$, and thus $A \not\delta_2 B$.

(10.8) Theorem. If $f : (X,\underset{\sim}{U}_1) \rightarrow (Y,\underset{\sim}{U}_2)$ is uniformly continuous, then $f : (X,\delta_1) \rightarrow (Y,\delta_2)$ is proximally continuous where $\delta_i = \delta(\underset{\sim}{U}_i)$ for $i = 1,2$.

Proof. Suppose on the contrary that $A \delta_1 B$, but $f(A) \not\delta_2 f(B)$. Then there exists a $U_2 \in \underset{\sim}{U}_2$ such that $(f(A) \times f(B)) \cap U_2 = \emptyset$. Since f is uniformly continuous, there exists a $U_1 \in \underset{\sim}{U}_1$ such that $(x,y) \in U_1$ implies $(f(x),f(y)) \in U_2$. But $A \delta_1 B$, so that $(A \times B) \cap U_1 \neq \emptyset$ implies $(f(A) \times f(B)) \cap U_2 \neq \emptyset$, a contradiction.

(10.9) Remarks. The converse of the above theorem is not true. Consider the identity mapping $i : (X,\underset{\sim}{U}_2) \rightarrow (X,\underset{\sim}{U}_1)$ where X, $\underset{\sim}{U}_1$ and $\underset{\sim}{U}_2$ are defined as in 10.4 (ii). Then i is a proximity mapping from (X,δ) onto itself, but i is not uniformly continuous. However, a proximity mapping is uniformly continuous under certain conditions: for example, when the uniformity of the domain space is pseudo-metrizable, or that of the range space is totally bounded. Moreover, given a proximity mapping $f : (X,\delta_1) \rightarrow (Y,\delta_2)$ and a uniformity $\underset{\sim}{U}_2$ on Y such that $\delta_2 = \delta(\underset{\sim}{U}_2)$, there exists a uniformity $\underset{\sim}{U}_1$ on X such that $\delta_1 = \delta(\underset{\sim}{U}_1)$ and $f : (X,\underset{\sim}{U}_1) \rightarrow (Y,\underset{\sim}{U}_2)$ is uniformly continuous.

§11. Completion of a Uniform Space by Cauchy Clusters

Let $(X,\underset{\sim}{U})$ be a uniform space and $\delta = \delta(\underset{\sim}{U})$. We have seen (Theorem (5.8)) that every ultrafilter in a proximity space generates a cluster and that given a set A in a cluster σ, there exists an ultrafilter containing A which generates σ. It therefore seems natural to call a cluster *Cauchy* if it is generated by a Cauchy ultrafilter. One can readily convince oneself that every Cauchy cluster is a *point cluster*, i.e. one which is determined by a point of X or of its completion. The neighbourhood system of the point will contain arbitrarily "small" sets which intersect every member of the cluster. This leads to the following definition of a Cauchy cluster, which is equivalent to the above (as will be seen later), but is easier to work with. Throughout this section, we shall suppose that $\underset{\sim}{U}$ is an open symmetric base for a uniformity.

(11.1) *Definition.* A cluster σ in $(X,\underset{\sim}{U},\delta)$ is *Cauchy* iff there exists a filter $\underset{\sim}{M} \subset \sigma$ such that for every $U \in \underset{\sim}{U}$, there exists an $M \in \underset{\sim}{M}$ such that $M \times M \subset U$ and $M \cap C \neq \emptyset$ for every $C \in \sigma$

(11.2) *Remarks.* (i) Every point cluster in X is Cauchy.

(ii) If a cluster is Cauchy, then every ultrafilter which generates it is Cauchy.

The following definition is easily seen to be equivalent to the usual one.

(11.3) Definition. A uniform space $(X,\underset{\sim}{U})$ is complete iff every Cauchy cluster in $(X,\underset{\sim}{U},\delta)$ is a point cluster σ_x for some $x \in X$.

(11.4) Lemma. A closed subspace Y of a complete uniform space $(X,\underset{\sim}{U})$ is complete.

Proof. The trace $\underset{\sim}{U}_Y = \{U \cap Y \times Y : U \in \underset{\sim}{U}\}$ of $\underset{\sim}{U}$ on Y is a base for the subspace uniformity on Y. If σ_1 is any Cauchy cluster in Y, then a slight modification of the proof of Theorem (5.17) shows that σ_1 is a subclass of a unique Cauchy cluster σ_2 in X. Since X is complete, $\sigma_2 = \sigma_x$ for some $x \in X$. But $x \, \delta \, B$ for every $B \in \sigma_1$ and since Y is closed, $x \in Y$. Therefore $\{x\} \in \sigma_1$.

Let f be a mapping which associates with each point $x \in X$, the point cluster σ_x. Then f is a one-to-one mapping of X onto the space $f(X)$ of all point clusters. Let X^* denote the set of all Cauchy clusters in X. From 11.2 (i), it follows that $f(X) \subset X^*$.

Let $\underset{\sim}{M}(\sigma)$ be a filter in σ such that

(i) $M \cap C \neq \emptyset$ for every $M \in \underset{\sim}{M}$ and every $C \in \sigma \in X^*$,

and

(ii) For every $U \in \underset{\sim}{U}$, there exists an $M \in \underset{\sim}{M}$ such that $M \times M \subset U$.

For each $U \in \underset{\sim}{U}$, define

$$U^* = \{(\sigma_1,\sigma_2) \in X^* \times X^* : \text{there exists } M \in \underset{\sim}{M}(\sigma_1), N \in \underset{\sim}{M}(\sigma_2) \text{ such that } M \times N \subset U\}.$$

(11.5) *Lemma*. $\underset{\sim}{U}^* = \{U^* : U \in \underset{\sim}{U}\}$ is a uniformity base on X^*.

Proof. Every U^* obviously contains the diagonal, and $(U \cap V)^* \subset U^* \cap V^*$. Given $U^* \in \underset{\sim}{U}^*$, there exists a $V \in \underset{\sim}{U}$ such that $V \circ V \subset U$. That $V^* \circ V^* \subset U^*$ follows from the following argument: if $(\sigma_1, \sigma_2) \in V^* \circ V^*$, then there exists a $\sigma_3 \in X^*$ such that $(\sigma_1, \sigma_3) \in V^*$ and $(\sigma_3, \sigma_2) \in V^*$. Hence there exists an $A \in \underset{\sim}{M}(\sigma_1)$, $B \in \underset{\sim}{M}(\sigma_2)$ and $C', C'' \in \underset{\sim}{M}(\sigma_3)$ such that $A \times C' \subset V$ and $C'' \times B \subset V$. Letting $C = C' \cap C'' \in \underset{\sim}{M}(\sigma_3)$, we have $A \times C \subset V$ and $C \times B \subset V$. Hence $A \times B \subset V \circ V \subset U$, which implies $(\sigma_1, \sigma_2) \in U^*$.

(11.6) *Lemma*. $f(X)$ is a dense subset of X^*.

Proof. Given any $\sigma \in X^*$ and any $U^* \in \underset{\sim}{U}^*$, we must show that $(f(X) \times \sigma) \cap U^* \neq \emptyset$. Choose $V \in \underset{\sim}{U}$ such that $V \circ V \circ V \subset U$. Since σ is Cauchy, there exists an $M \in \underset{\sim}{M}(\sigma)$ since that $M \times M \subset V$. Then $V[M] \times V[M] \subset U$. Since $V[M]$ is open (recall V is open and symmetric), we can choose a point $p \in V[M]$ and a $W \in \underset{\sim}{U}$ such that $W[p] \subset V[M]$. Then $W[p] \times W[p] \subset U$, where $W[p] \in \underset{\sim}{M}(\sigma_p)$ and $V[M] \in \underset{\sim}{M}(\sigma)$. Therefore $(\sigma_p, \sigma) \in U^*$ where $\sigma_p \in f(X)$.

Let δ^* be the natural proximity induced by $\underset{\sim}{U}^*$ on X^*. The restriction $\underset{\sim}{U}^*_f$ of U^* to $f(X)$ is a uniformity base on $f(X)$ and so induces the natural proximity δ^*_f on $f(X)$.

(11.7) Lemma. $(X,\underset{\sim}{U},\delta)$ and $(f(X),U_f^*,\delta_f^*)$ are proximally isomorphic.

Proof. Clearly f is one-to-one and onto. Suppose $A \delta B$. Given U_f^*, we must show that $(f(A) \times f(B)) \cap U_f^* \neq \emptyset$. Now there exists $a \in A$, $b \in B$ such that $(a,b) \in V$. Then $V[a] \times V[b] \subset U$ and the point clusters σ_a and σ_b are such that $(\sigma_a,\sigma_b) \in U_f^*$. On the other hand, if $f(A)\ \delta_f^*\ f(B)$, then for each U_f^* there exists a $(\sigma_a,\sigma_b) \in U_f^*$ where $\sigma_a \in f(A)$ and $\sigma_b \in f(B)$. So $(a,b) \in U$, which means that $(A \times B) \cap U \neq \emptyset$ for all $U \in \underset{\sim}{U}$.

It may similarly be proved that $(X,\underset{\sim}{U})$ and $(f(X),\underset{\sim}{U}_f^*)$ are uniformly isomorphic.

(11.8) Lemma. Every Cauchy cluster in $(X^*,\underset{\sim}{U}^*,\delta^*)$ has a point.

Proof. Let σ^* be any Cauchy cluster in X^*. Since $f(X)$ is dense in X^*, a slight modification to the proof of Theorem (5.16) shows that σ^* determines a unique Cauchy cluster σ' in $f(X)$ such that $\sigma' \subset \sigma^*$. But σ' is isomorphic to a Cauchy cluster σ in X. In order to show that $\sigma \in \sigma^*$, it is sufficient to verify that for each $U^* \in \underset{\sim}{U}$ and each $M \in \sigma'$, $(\sigma \times M) \cap U^* \neq \emptyset$. Given $U^* \in \underset{\sim}{U}^*$, there exists a $V \in \underset{\sim}{U}$ and $C \in \underset{\sim}{M}(\sigma)$ such that $V \circ V \circ V \subset U$ and $C \times C \subset V$. Then $V[C] \times V[C] \subset U$. Letting $M_o = V[C] \cap f^{-1}(M)$ we have $M_o \in \sigma$ since $V[C] \in \underset{\sim}{M}(\sigma)$, $f^{-1}(M) \in \sigma$, and we can find an ultrafilter containing both

$V[C]$ and $f^{-1}(M)$ which generates σ. Choose a point $p \in M_o$. Since $V[C]$ is open, there exists a $W \in \underset{\sim}{U}$ such that $W[p] \subset V[C]$. Hence we have $W[p] \times V[C] \subset U$ where $W[p] \in \underset{\sim}{M}(\sigma_p)$, $V[C] \in \underset{\sim}{M}(\sigma)$ and $\sigma_p \in M$. Thus $(\sigma_p, \sigma) \in U^*$ and $(\sigma \times M) \cap U^* \neq \emptyset$.

The above result shows that $(X^*, \underset{\sim}{U}^*, \delta^*)$ is complete; for if $\underset{\sim}{F}$ is any Cauchy filter in X^*, then $\underset{\sim}{F}$ is contained in a Cauchy ultrafilter. This ultrafilter generates a Cauchy cluster which, by Lemma (11.8), must be a point cluster σ_{x_o} for some $x_o \in X^*$. Clearly x_o is a cluster point of the Cauchy filter $\underset{\sim}{F}$, and thus $\underset{\sim}{F}$ converges to x_o.

Finally, we remark that every Cauchy cluster in X is generated by the Cauchy ultrafilter containing the neighbourhood filter of a point which is either in X or in X^*. (If the point is in X^*, consider the trace of its neighbourhood filter on X.) Hence the Definition (11.1) is equivalent to: a cluster is Cauchy iff it is generated by a Cauchy ultrafilter.

NOTES

§10. The definition of a uniformity, as given in this section, is due to Weil [K]. That every such uniformity has a base consisting of open symmetric members is proved on page 179 of Kelley [H]. Smirnov [88] was the first to investigate the relationships between proximity spaces and uniform spaces, although he worked with uniform structures defined by systems of uniform coverings. This equivalent method of defining a uniformity was earlier employed by Tukey [J]. Further work concerning the connection between uniformity and proximity was carried out by Gál [25] and by Alfsen and Fenstad [4] using Weil's uniform structures. Ramm gave the first example of two different uniform structures inducing the same proximity.

§11. This section owes its existence to the work of Warrack and Naimpally [106]. Definition (11.3) is equivalent to the usual one; namely, a uniform space is complete iff every Cauchy filter in the space converges to some point of the space.

GENERAL REFERENCES

A. Alexandroff, P. (1939), On bicompact extensions of topological spaces, Mat. Sb., 5(47), 403-423 (in Russian; German summary).

B. Freudenthal, H. (1942), Neuaubau der Endentheorie, Ann. of Math., (2)43, 261-279.

C. Gaal, S.A. (1964) Point set topology, Academic Press, New York.

D. Gál, I.S. (1959), Uniformizable spaces with a unique structure, Pacific J. Math., 9, 1053-1060.

E. Gillman, L. and M. Jerison (1960), Rings of continuous functions, Van Nostrand, Princeton.

F. Hu, S.-T. (1949), Boundedness in a topological space, J. Math. Pures Appl., 28, 287-320.

G. Iliadis, S. and S. Fomin (1966), The method of centered systems in the theory of topological spaces, Uspekhi Math. Nauk, 21, 47-76 (in Russian); English translation in Russian Math. Surveys, 21, 37-62.

H. Kelley, J.L. (1961), General topology, Van Nostrand, Princeton.

I. Magill, K.D., Jr. (1965), N-point compactifications, Amer. Math. Monthly, 72, 1075-1081.

J. Tukey, J.W. (1940), Convergence and uniformity in topology, Annals of Math. Studies 2, Princeton.

K. Weil, A. (1937), Sur les espaces a structure uniforme et sur la topologie générale, Actualités Sci. Ind., 551, Paris.

BIBLIOGRAPHY FOR PROXIMITY SPACES

1. Aarts, J.M. (1966), Dimension and deficiency in general topology, Druk. V.R.B., Kleine der A 3-4 Groningen.

2. Alexandroff, P. (1954), Aus der Mengentheoretischen Topologie der letzten zwanzig Jahren, Proc. of the International Congress of Mathematicians, Amsterdam, I, 177-196.

3. Alexandroff, P. (1956), On two theorems of Yu. Smirnov in the theory of bicompact extensions, Fund. Math., 43, 394-398 (in Russian).

4. Alfsen, E.M. and J. E. Fenstad (1959), On the equivalence between proximity structures and totally bounded uniform structures, Math. Scand., 7, 353-360 (Correction (1961), Ibid, 9, 258).

5. Alfsen, E.M. and J. E. Fenstad (1960), A note on completion and compactification, Math. Scand., 8, 97-104.

6. Alfsen, E.M. and O. Njåstad (1963), Proximity and generalized uniformity, Fund. Math., 52, 235-252.

7. Alfsen, E.M. and O. Njåstad (1963), Totality of uniform structures with linearly ordered base, Fund. Math., 52, 253-256.

8. Banaschewski, Bernhard and Jean-Marie Maranda (1961), Proximity functions, Math. Nachr., 23, 1-37.

9. Bognár, M. (1962), Bemerkungen zum Kongressvortrag "Stetigkeit und abstrakte Mengenlehre", von F. Riesz, General Topology and Its Relations to Modern Analysis and Algebra, Praha.

10. Čech, Eduard (1966), Topological Spaces, Interscience Publishers, John Wiley and Sons.

11. Császár, A. (1957), Sur une classe de structures topologiques générales, Rev. Math. Pures Appl., 2, 399-407.

12. Császár, A. (1963), Foundations of General Topology, Macmillan, New York.

13. Császár, A. et S. Mrówka (1959), Sur la compactification des espaces de proximité, Fund. Math., 46, 195-207.

14. Dočinov, D. (1964), A method of introducing the concept of proximity, C. R. Acad. Bulgare Sci., 17, 349-351 (in Russian).

15. Dočinov, D. (1964), A unified theory of topological spaces, proximity spaces and uniform spaces, Dokl. Akad. Nauk SSSR, 156, 21-24 (in Russian).

16. Dowker, C.H. (1961), Mappings of proximity structures, Proc. Symp. Gen. Top., Prague, 139-141.

17. Efremovič, V.A. (1951), Proximity spaces, Dokl. Akad. Nauk SSSR, 76, 341-343 (in Russian).

18. Efremovič, V.A. (1952), The geometry of proximity, Mat. Sb., 31 (73), 189-200 (in Russian).

19. Efremovič, V.A. and A.S. Švarc (1953), A new definition of uniform spaces. Metrization of proximity spaces, Dokl. Akad. Nauk SSSR, 89, 393-396 (in Russian).

20. Efremovič, V.A.; Mordkovič, A.G. and V.J. Sandberg (1967), Correct spaces, Dokl. Akad. Nauk SSSR, 172, 1254-1257 (in Russian); English translation in Soviet Math. Dokl., 8, No. 1, 254-258.

21. Fenstad, J.E. (1964), On ℓ-groups of uniformly continuous functions. III. Proximity spaces, Math. Z., 83, 133-139.

22. Fomin, S.V. (1958), On the connection between proximity spaces and the bicompact extension of completely regular spaces, Dokl. Akad. Nauk SSSR, 121, 236-238 (in Russian).

23. Gacsályi, Sándor (1964), On proximity functions and symmetrical topogenous structures, Publ. Math. Debreceu, 11, 165-174.

24. Gacsályi, Sándor (1965), Skew proximity functions, Publ. Math. Debrecen, 12, 271-280.

25. Gál, I.S. (1959), Proximity relations and precompact structures, Indag. Math. 21, 304-326.

26. Goetz, A. (1962), On a notion of uniformity for L-spaces of Fréchet, Colloq. Math., 9, 223-231.

27. Goetz, A. (1967), Proximity-like structures in convergence spaces; (Presented at conference on General Topology at Arizona State Univ.).

28. Hacque, M. (1962), Sur les E-structures, C. R. Acad. Sci. Fr., 254, 1905-1907.

29. Haddad, L. (1962), Sur la notion de tramail, C. R. Acad. Sci. Fr., 255, 2880-2882.

30. Hayashi, E. (1964), On some properties of a proximity, J. Math. Soc. Japan, 16, 375-378.

31. Hayashi, E. (1965), A note on proximity spaces, Bull. Aichi Gakngei Univ., 14, 1-4.

32. Heuchenne, C. (1965), Sur les connexions entre topologies et proximités, Bull. Soc. r. Sci. Liege, 34, 215-230.

33. Hursch, Jack L., Jr. (1965), Proximity and height, Math. Scand., 17, 150-160.

34. Hursch, Jack L., Jr. (1966), An example of two uniformities equal in height and proximity, (unpublished).

35. Hušek, M. (1964), S-categories, Comment. Math. Univ. Carolinae, 5, 37-46.

36. Hušek, M. (1964), Generalized proximity and uniform spaces, Comment. Math. Univ. Carolinae, 5, 247-266.

37. Hušek, M. (1965), Generalized proximity and uniform spaces II, Comment. Math. Univ. Carolinae, 6, 119-139.

38. Isbell, J.R. (1960), On inductive dimension of proximity spaces, Dokl. Akad. Nauk SSSR, 134, 36-38 (in Russian); English translation in Soviet Math. Dokl., 1, 1013-1015.

39. Isbell, J.R. (1964), Uniform Spaces, Mathematical Surveys, No. 12, Amer. Math. Soc., Prov., R.I.

40. Ivanov, A.A. (1959), Proximity relations on topological spaces, Dokl. Akad. Nauk SSSR, 128, 33-36 (in Russian).

41. Ivanova, V.M. (1964), Proximity relations and spaces of close subsets, Math. Sb., 65(107), 18-27 (in Russian).

42. Ivanova, V.M. and A.A. Ivanov (1959), Proximity spaces and bicompact extensions of topological spaces, Dokl. Akad. Nauk SSSR, 127, 20-22 (in Russian).

43. Jarulkin, N.G. (1956), Generalized infinitesimal spaces, Ivanov. Gos. Ped. Inst. Uč. Zap. Fiz.-Mat. Nauki., 10, 61-79 (in Russian).

44. Jarulkin, N.G. (1957), On the generalized proximity spaces, Math. Sb., 43(85), 397-400 (in Russian).

45. Katětov, M. (1960), Über die Berührungsräume, Wiss. Z. Humboldt-Univ. Berlin math.-natur. R. 9, 685-691 (Russian, English and French summaries).

46. Katětov, M. and J. Vanicek (1964), On the proximity generated by entire functions, Comment. Math. Univ. Carolinae, 5, 267-278.

47. Krishna Murti, S.B. (1940), A set of axioms for topological algebra, J. Indian Math. Soc., 4, 116-119.

48. Leader, S. (1959), On clusters in proximity spaces, Fund. Math., 47, 205-213.

49. Leader, S. (1960), On completion of proximity spaces by local clusters, Fund. Math., 48, 201-216.

50. Leader, S. (1962), On duality in proximity spaces, Proc. Amer. Math. Soc., 13, 518-523.

51. Leader, S. (1963), On a problem of Alfsen and Fenstad, Math. Scand., 13, 44-46.

52. Leader, S. (1964), On products of proximity spaces, Math. Annalen, 154, 185-194.

53. Leader, S. (1965), On pseudometrics for generalized uniform structures, Proc. Amer. Math. Soc., 16, 493-495.

54. Leader, S. (1967), Local proximity spaces, Math. Annalen, 169, 275-281.

55. Leader, S. (1967), Spectral structures and uniform continuity, Fund. Math., 60, 105-115.

56. Lodato, M.W. (1964), On topologically induced generalized proximity relations, Prox. Amer. Math. Soc., 15, 417-422.

57. Lodato, M.W. (1966), On topologically induced generalized proximity relations II, Pacific Jour. Math., 17, 131-136.

58. Lodato, M.W. (1966), Generalized proximity spaces: A generalization of topological spaces, (unpublished).

59. Mamuzič, Z.P. (1963), Introduction to General Topology, P. Noordhoff Ltd., Groningen, The Netherlands.

60. Mattson, D.A. (1967), Separation relations and quasi-proximities, Math. Annalen, 171, 87-92.

61. Meenakshi, K.N. (1966), Proximity structures in Boolean algebras, Acta Sci. Math. (Szeged), 27, 85-92.

62. Mordkovič, A.G. (1965), Systems with small sets and proximity spaces, Mat. Sb., 67(109), 474-480 (in Russian).

63. Mordkovič, A.G. (1966), Test for correctness of a uniform space, Dokl. Akad. Nauk SSSR, 169, 276-279 (in Russian); English translation in Soviet Math. Dokl., 7, 915-918.

64. Mrówka, S.G. (1956), On the notion of competeness in proximity spaces, Bull. Acad. Polon. Sci. Cl. III, 4, 477-478.

65. Mrówka, S.G. (1956), On complete proximity spaces, Dokl. Akad. Nauk SSSR, 108, 587-590 (in Russian).

66. Mrówka, S.G. (1957), On the uniform convergences in proximity spaces, Bull. Acad. Polon. Sci. Cl. III, 5, 255-257.

67. Mrówka, S.G. (1960), Axiomatic characterization of the family of all clusters in a proximity space, Fund. Math., 48, 123-126.

68. Mrówka, S.G. and W. J. Pervin (1964), On uniform connectedness, Proc. Amer. Math. Soc., 15, 446-449.

69. Myškis, A.D. (1952), On the relation of infinitesimal spaces, Dokl. Akad. Nauk SSSR, 84, 879-882 (in Russian).

70. Myškis, A.D. and E. I. Vigant (1955), On a connection of proximity spaces with extensions of topological spaces, Dokl. Akad. Nauk SSSR, 103, 969-972 (in Russian).

71. Naimpally, S.A. and B.D. Warrack (1967), On generalized proximities, (unpublished).

72. Njåstad, O. (1963), Some properties of proximity and generalized uniformity, Math. Scand., 12, 47-56.

73. Njåstad, O. (1964), On real-valued proximity mappings, Math. Annalen, 154, 413-419.

74. Njåstad, O. (1965), On uniform spaces where all uniformly continuous functions are bounded, Monatsh. Math., 69, 167-176.

75. Njåstad, O. (1966), On Wallman-type compactifications, Math. Z., 91, 267-276.

76. Pervin, W.J. (1963), Quasi-proximities for topological spaces, Math. Annalen, 150, 325-326.

77. Pervin, W.J. (1964), On separation and proximity spaces, Amer. Math. Monthly, 71, 158-161.

78. Pervin, W.J. (1964), Equinormal proximity spaces, Indag. Math., 26, No. 2, 152-154.

79. Poljakov, V.Z. (1964), Regularity, products and spectra of proximity spaces, Dokl. Akad. Nauk SSSR, 154, 51-54 (in Russian); English translation in Soviet Math. Dokl., 5, 45-49.

80. Poljakov, V.Z. (1964), Open mappings of proximity spaces, Dokl. Akad. Nauk SSSR, 155, 1014-1017 (in Russian).

81. Poljakov, V.Z. (1965), Regularity and the product of proximity spaces, Mat. Sb., 67(109), 428-439 (in Russian).

82. Poljakov, V.Z. (1965), On the regularity of the proximity product of regular spaces, Mat. Sb., 68(110), 242-250 (in Russian).

83. Poljakov, V.Z. (1965), On uniform convergence spaces, Colloquium Math., 13, 167-179 (in Russian).

84. Ramm, N.S. and A.S. Švarc (1953), Geometry of proximity, uniform geometry and topology, Mat. Sb., 33(75), 157-180 (in Russian).

85. Riesz, F. (1908), Stetigkeitsbegriff und abstrakete mengenlehre, Atti IV Congr. Intern. Mat. Roma, II, 18-24.

86. Sieber, J.L. and W.J. Pervin (1964), Connectedness in syntopogenous spaces, Proc. Amer. Math. Soc., 15, 590-595.

87. Smirnov, Y.M. (1952), On proximity spaces in the sense of V. A. Efremovič, Dokl. Akad. Nauk SSSR, 84, 895-898; English translation in Amer. Math. Soc. Transl. Ser. 2, 38, 1-4.

88. Smirnov, Y.M. (1952), On proximity spaces, Mat. Sb., 31(73), 543-574; English translation in Amer. Math. Soc. Transl. Ser. 2, 38, 5-35.

89. Smirnov, Y.M. (1953), On the completeness of proximity spaces, Dokl. Akad. Nauk SSSR, 88, 761-764 (in Russian).

90. Smirnov, Y.M. (1953), On completeness of uniform spaces and proximity spaces. Dokl. Akad. Nauk SSSR, 91, 1281-1284 (in Russian).

91. Smirnov, Y.M. (1954), On the dimension of proximity spaces, Dokl. Akad. Nauk SSSR, 95, 717-720 (in Russian).

92. Smirnov, Y.M. (1956), On the dimension of proximity spaces, Mat. Sb., 38 (80), 283-302 (in Russian); English translation in Amer. Math. Soc. Transl. Ser. 2, 21, 1-20.

93. Smirnov, Y.M. (1956), Geometry of infinite uniform complexes and δ-dimensionality of point sets, Mat. Sb., 40(82), 137-156 (in Russian); English translation in Amer. Math. Soc. Transl. Ser. 2, 15, 95-113.

94. Smirnov, Y.M. (1962), Some questions of uniform topology, Proc. Internat. Congr. Mathematicians, Stockholm, 497-501.

95. Smirnov, Y.M. (1964), On the completeness of proximity spaces I, Trudy Moskov Mat. Obšč, 3, 271-306 (in Russian); English translation in Amer. Math. Soc. Transl. Ser. 2, 38, 37-73.

96. Smirnov, Y.M. (1964), On the completeness of proximity spaces II, Trudy Moskov Mat. Obšč, 4, 421-438 (in Russian); English translation in Amer. Math. Soc. Transl. Ser. 2, 38, 75-93.

97. Smirnov, Y.M. (1966), On the dimension of remainders in bicompact extensions of proximity and topological spaces, Mat. Sb., 69(111), 141-160 (in Russian).

98. Smith, D. Hammond (1966), Hyperspaces of a uniformizable space, Proc. Cambridge Philos. Soc., 62, 25-28.

99. Steiner, E.F. (1964), The relation between quasi-proximities and topological spaces, Math. Annalen, 155, 194-195.

100. Svarc, A.S. (1956), Proximity spaces and lattices, Učen. Zap. Ivanovsk Gos. Ped. Inst., 10, 55-60 (in Russian).

101. Szymanski, P. (1941), La notion des ensembles séparés terme primitif de la topologie, Math. Timisoara, 17, 65-84.

102. Thron, W.J. (1966), Topological Structures, Holt, Rinehart and Winston, Toronto.

103. Wallace, A.D. (1941), Separation spaces, Annals of Math., 42, 687-697.

104. Wallace, A.D. (1942), Separation spaces II, Anais. Acad. Brosileira Ciencias, 14, 203-206.

105. Warrack, B.D. and S.A. Naimpally (1966), Clusters and ultrafilters, (to be published in Mate. Vesnik).

106. Warrack, B.D. and S.A. Naimpally (1967), On completion of a uniform space by Cauchy clusters, (unpublished).

B29876